THE UNIVERSITY
OF ILLINOIS
LIBRARY
510.5
QU
v.19
READING ROOM
MATHEMATICS LIBRARY

The person charging this material is responsible for its return to the library from which it was withdrawn on or before the **Latest Date** stamped below.

Theft, mutilation, and underlining of books are reasons for disciplinary action and may result in dismissal from the University.

UNIVERSITY OF ILLINOIS LIBRARY AT URBANA-CHAMPAIGN

APR 26 1978

APR 14 RECD

JUN 23 1978

JUN 23 RECD

L161—O-1096

Digitized by the Internet Archive
in 2021 with funding from
University of Illinois Urbana-Champaign

https://archive.org/details/quarterlyjournal1918unse

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

N. M. FERRERS, D.D., F.R.S.,
MASTER OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,
SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE; AND

J. W. L. GLAISHER, M.A., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

UNIVERSITY OF ILLINOIS LIBRARY

VOL. XIX.

London:
LONGMANS AND CO.
PATERNOSTER ROW.

1883.

Cambridge:

W. METCALFE AND SON, TRINITY STREET AND ROSE CRESCENT.

510.5
Qu
v.19

CONTENTS TO VOL. XIX.

PAGE

On Curvilinear Coordinates. By Prof. *A. Cayley* - - - - 1

On certain Formulæ in Elliptic Functions. By *J. W. L. Glaisher* - - 22

Note on the Standard Solutions of a System of Linear Equations. By Prof. *A. Cayley* - - - - - - - - - 38

Sur une Application d'un Determinant. Par Prof. *A. Legoux* - - 41

On Spherical Cycloidal and Trochoidal Curves. By *Henry M. Jeffery* - 44

On the Forces experienced by a Solid moving in an Infinite Mass of Liquid. By *H. Lamb* - - - - - - - - - 66

Expressions for arg sn a and $(\arg \operatorname{sn} a)^2$ as Definite Integrals. By *J. W. L. Glaisher* - - - - - - - - - 71

On Systems of Circles and Bicircular Quartics. By *Homersham Cox* 74

Note on a Differential Equation due to Kummer. By Prof. *A. R. Forsyth* - 125

On Seminvariants. By Prof. *Cayley* - - - - - - 131

Expression for the Area of a Convex Quadrilateral when the sum of two opposite Angles is given. By *A. H. Anglin* - - - - 138

Note on the Energy of Strain of an Isotropic Solid. By *H. T. Stearn* - 140

Note on the Time of Descent down the Arc of a Vertical Parabola. By *J. W. L. Glaisher* - - - - - - - - - 141

A System of Integrals involving Elliptic Functions. By *J. W. L. Glaisher* - 145

On the Differential Equation $X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0$. By Captain *P. A. MacMahon* - - - - - - - - 158

Note on the foregoing Paper. By Prof. *Cayley* - - - - 182

Note on the Spherical Triangle of Elliptic Functions. By Prof. *W. W. Johnson* - - - - - - - - - - 185

Notice. *The Acta Mathematica* - - - - - - 189

On the Motion of a Liquid in and about Cylinders whose Transverse Sections are the Inverse of Confocal Ellipses with respect to their Centres. By *A. B. Basset* - - - - - - - - - 190

PAGE

Note on the Compositions of a Number as a Sum of Two and Four Uneven Squares. By *J. W. L. Glaisher* - - - - - - 212

Theorems relating to the Sum of the Uneven Divisors of a Number. By *J. W. L. Glaisher* - - - - - - - - - 216

On Mr. Anglin's Formula for the successive Powers of the Root of an Algebraical Equation. By Prof. *Cayley* - - - - - 223

A revision of Chapters XXIV. and XXVI. of Legendre's Fonctions Elliptiques, t. I. By *A. G. Greenhill* - - - - - - 225

On certain Definite Integrals connected with Spherical Harmonics. By *Percival Frost* - - - - - - - - - - 242

On the Solution of the General Equation of the Fourth Degree. By *A. Lodge* - - - - - - - - - - - 257

A Method of expressing any particular Arbitrary Constant in the Solution of Linear Differential Equations in Terms of the Initial Conditions. By *E. J. Routh* - - - - - - - - - - 262

On Linear Differential Equations: in particular that satisfied by the series $1+\frac{\alpha\beta\theta}{\gamma\varepsilon}x+\frac{\alpha.\alpha+1.\beta.\beta+1.\theta.\theta+1}{1.2.\gamma.\gamma+1.\varepsilon.\varepsilon+1}x^2+\ldots$. By *A. R. Forsyth* - 292

On Professor Cayley's Canonical Form. By Captain *P. A. Mac Mahon* - 337

The "Triplicate-Ratio" Circle. By *R. Tucker* - - - - - 342

On certain Physical Problems connected with Surfaces which are the Inverses of Ellipsoids of Revolution. By *A. B. Basset* - - - - - 349

On some Abelian Integrals. By Prof. *Rink* - - - - - - 374

On Lines of Striction. By *J. Larmor* - - - - - - 381

WITH PLATES I., II.

UNIVERSITY OF ILLINOIS LIBRARY

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED MATHEMATICS.

ON CURVILINEAR COORDINATES.

By Professor A. CAYLEY.

THE present Memoir is based upon Mr. Warren's "Exercises in Curvilinear and Normal Coordinates," *Camb. Phil. Trans.* t. XII (1877), pp. 455–502, and has for a principal object the establishment of the six differential equations of the second order corresponding to his six equations for normal coordinates: but the notation is different; the results are more general, inasmuch as I use throughout general curvilinear coordinates instead of his normal coordinates; and as regards my six equations for general curvilinear coordinates, the terms containing differential coefficients of the first order are presented under a different form.

If the position of a point in space is determined by the rectangular coordinates x, y, z; then p, q, r being each of them a given function of x, y, z, we have conversely x, y, z, each of them a given function of p, q, r, which are thus in effect coordinates serving to determine the position of the point, and are called curvilinear coordinates.

But it is not in the first instance necessary to regard x, y, z as rectangular coordinates, or even as Cartesian coordinates at all; we are simply concerned with the two sets of variables x, y, z and p, q, r, each variable of the one set being a given function of the variables of the other set; and in particular the x, y, z are regarded as being each of them a given function of the p, q, r.

Except as regards the symbols ξ, η, ζ presently mentioned, the suffixes 1, 2, 3 refer to the variables p, q, r respectively, and are used to denote differentiations in regard to these variables, viz.

$$x_1,\ x_2,\ x_3,\ x_{11},\quad x_{12},\ x_{22},\ldots$$

are written to denote

$$\frac{dx}{dp},\ \frac{dx}{dq},\ \frac{dx}{dr},\ \frac{d^2x}{dp^2},\ \frac{d^2x}{dp\,dq},\ \frac{d^2x}{dq^2},\ \&c.,$$

and so in other cases; in particular

$$x_1,\ x_2,\ x_3 \text{ denote } \frac{dx}{dp},\ \frac{dx}{dq},\ \frac{dx}{dr},$$

$$y_1,\ y_2,\ y_3 \quad ,, \quad \frac{dy}{dp},\ \frac{dy}{dq},\ \frac{dy}{dr},$$

$$z_1,\ z_2,\ z_3 \quad ,, \quad \frac{dz}{dp},\ \frac{dz}{dq},\ \frac{dz}{dr}.$$

The minors formed with these differential coefficients are denoted by suffixed letters ξ, η, ζ, thus

$$\xi_1,\ \xi_2,\ \xi_3 \text{ denote } y_2z_3 - y_3z_2,\ y_3z_1 - y_1z_3,\ y_1z_2 - y_2z_1,$$

$$\eta_1,\ \eta_2,\ \eta_3 \quad ,, \quad z_2x_3 - z_3x_2,\ z_3x_1 - z_1x_3,\ z_1x_2 - z_2x_1,$$

$$\zeta_1,\ \zeta_2,\ \zeta_3 \quad ,, \quad x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1,$$

so that, as regards these letters ξ, η, ζ, the suffixes do *not* denote differentiations.

The determinant $\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$ is put $= L$,

and the symbols (a, b, c, f, g, h), (A, B, C, F, G, H) are defined as follows:

$$\mathrm{a} = x_1^2 + y_1^2 + z_1^2, \qquad \mathrm{A} = \xi_1^2 + \eta_1^2 + \zeta_1^2,$$

$$\mathrm{b} = x_2^2 + y_2^2 + z_2^2, \qquad \mathrm{B} = \xi_2^2 + \eta_2^2 + \zeta_2^2,$$

$$\mathrm{c} = x_3^2 + y_3^2 + z_3^2, \qquad \mathrm{C} = \xi_3^2 + \eta_3^2 + \zeta_3^2,$$

$$\mathrm{f} = x_2x_3 + y_2y_3 + z_2z_3, \qquad \mathrm{F} = \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3,$$

$$\mathrm{g} = x_3x_1 + y_3y_1 + z_3z_1, \qquad \mathrm{G} = \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1,$$

$$\mathrm{h} = x_1x_2 + y_1y_2 + z_1z_2, \qquad \mathrm{H} = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2,$$

we have then further

$$\left|\begin{matrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{matrix}\right| = L, \quad \left|\begin{matrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{matrix}\right| = L^2,$$

$$\left|\begin{matrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{matrix}\right| = L^2, \quad \left|\begin{matrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{matrix}\right| = \frac{1}{L^2},$$

$(A, B, C, F, G, H) =$

$(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$

$L^2(a, b, c, f, g, h) =$

$(BC - F^2,\ CA - G^2,\ AB - H^2,\ GH - AF,\ HF - BG,\ FG - CH),$

which equations are at once proved, and are fundamental ones in the theory.

It is convenient to add that we have

$$\left(\begin{matrix} \frac{dp}{dx}, & \frac{dp}{dy}, & \frac{dp}{dz} \\ \frac{dq}{dx}, & \frac{dq}{dy}, & \frac{dq}{dz} \\ \frac{dr}{dx}, & \frac{dr}{dy}, & \frac{dr}{dz} \end{matrix}\right) = \frac{1}{L} \left(\begin{matrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{matrix}\right),$$

that is $\frac{dp}{dx} = \frac{1}{L}\xi_1$, &c.; and further

$$dx^2 + dy^2 + dz^2 = (a, b, c, f, g, h)(dp, dq, dr)^2,$$

$$dp^2 + dq^2 + dr^2 = \frac{1}{L^2}(A, B, C, F, G, H)(dx, dy, dz)^2.$$

Differentiating the values of a, b, c, f, g, h, we have $\frac{1}{2}a_1 = x_1x_{11} + y_1y_{11} + z_1z_{11}$, which may be written in the abbreviated form $\frac{1}{2}a_1 = 1.11$; similarly

$$f_1 = x_2x_{13} + y_2y_{13} + z_2z_{13} + x_3x_{12} + y_3y_{12} + z_3z_{12},$$

which in like manner may be written $f_1 = 2.13 + 3.12$, and so in other cases; observe that in the duad part of any symbol the order of the numbers is immaterial, $2.13 = 2.31$. The whole system of equations is

$$\begin{matrix} \frac{1}{2}a_1 = 1.11, & \frac{1}{2}a_2 = 1.12, & \frac{1}{2}a_3 = 1.13, \\ \frac{1}{2}b_1 = 2.12, & \frac{1}{2}b_2 = 2.22, & \frac{1}{2}b_3 = 2.23, \\ \frac{1}{2}c_1 = 3.13, & \frac{1}{2}c_2 = 3.23, & \frac{1}{2}c_3 = 3.33, \\ f_1 = 2.13 + 3.12, & f_2 = 2.23 + 3.22, & f_3 = 2.33 + 3.23, \\ g_1 = 3.11 + 1.13, & g_2 = 3.12 + 1.23, & g_3 = 3.13 + 1.33, \\ h_1 = 1.12 + 2.11, & h_2 = 1.22 + 2.12, & h_3 = 1.23 + 2.13. \end{matrix}$$

These may also be written

$$1.11 = \tfrac{1}{2}a_1 \quad , \quad 1.22 = h_2 - \tfrac{1}{2}b_1 \quad , \quad 1.33 = g_3 - \tfrac{1}{2}c_1 \quad ,$$
$$2.11 = h_1 - \tfrac{1}{2}a_2 \quad , \quad 2.22 = \tfrac{1}{2}b_2 \quad , \quad 2.33 = f_3 - \tfrac{1}{2}c_2 \quad ,$$
$$3.11 = g_1 - \tfrac{1}{2}a_3 \quad , \quad 3.22 = f_2 - \tfrac{1}{2}b_3 \quad , \quad 3.33 = \tfrac{1}{2}c_3 \quad ,$$
$$1.23 = \tfrac{1}{2}(-f_1+g_2+h_3), \quad 1.31 = \tfrac{1}{2}a_3 \quad , \quad 1.12 = \tfrac{1}{2}a_2 \quad ,$$
$$2.23 = \tfrac{1}{2}b_3 \quad , \quad 2.31 = \tfrac{1}{2}(f_1-g_2+h_3), \quad 2.12 = \tfrac{1}{2}b_1 \quad ,$$
$$3.23 = \tfrac{1}{2}c_2 \quad , \quad 3.31 = \tfrac{1}{2}c_1 \quad , \quad 3.12 = \tfrac{1}{2}(f_1+g_2-h_3),$$

and it is to be observed that we can from each system of three equations express a set of second differential coefficients of the x, y, z in terms of the first differential coefficients of the a, b, c, f, g, h; thus the three equations containing 11, written at length, are

$$x_1.x_{11} + y_1.y_{11} + z_1.z_{11} = \tfrac{1}{2}a_1,$$
$$x_2 \;,, + y_2 \;,, + z_2 \;,, = h_1 - \tfrac{1}{2}a_2,$$
$$x_3 \;,, + y_3 \;,, + z_3 \;,, = g_1 - \tfrac{1}{2}a_3,$$

three linear equations for the determination of x_{11}, y_{11}, z_{11}; hence for x_{11} we have

$$x_{11} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$$
$$= \tfrac{1}{2}a_1 (y_2z_3 - y_3z_2) + (h_1 - \tfrac{1}{2}a_2)(y_3z_1 - y_1z_3) + (g_1 - \tfrac{1}{2}a_3)(y_1z_2 - y_2z_1),$$

or what is the same thing

$$Lx_{11} = \tfrac{1}{2}a_1.\xi_1 + (h_1 - \tfrac{1}{2}a_2)\,\xi_2 + (g_1 - \tfrac{1}{2}a_3)\,\xi_3,$$

and so for y_{11}, z_{11}; or if (as in the sequel) we desire the value of a linear function $\alpha x_{11} + \beta y_{11} + \gamma z_{11}$, calling this for a moment $\square$, we join to the foregoing a new equation

$$\alpha x_{11} + \beta y_{11} + \gamma z_{11} = \square,$$

then eliminating the three quantities we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \square \\ x_1, & y_1, & z_1, & \tfrac{1}{2}a_1 \\ x_2, & y_2, & z_2, & h_1 - \tfrac{1}{2}a_2 \\ x_3, & y_3, & z_3, & g_1 - \tfrac{1}{2}a_3 \end{vmatrix} = 0,$$

giving $L\square$ as a linear function of $\tfrac{1}{2}a_1$, $h_1 - \tfrac{1}{2}a_2$, $g_1 - \tfrac{1}{2}a_3$.

We can in like manner form the expressions for the second

differential coefficients of the a, b, c, f, g, h, these will of course contain third differential coefficients of the x, y, z.

Writing down only what is wanted, we have

$$\tfrac{1}{2}a_{22} = 12.12 + 1.122,$$

$$\tfrac{1}{2}b_{11} = 12.12 + 2.112,$$

$$h_{12} = 12.12 + 11.22 + 1.122 + 2.112$$

where of course 12.12 denotes $x^2_{12} + y^2_{12} + z^2_{12}$, 1.122 denotes $x_1.x_{122} + y_1.y_{122} + z_1 z_{122}$, and so in other cases: it follows that

$$\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12}) = 12.12 - 11.22,$$

so that the third differential coefficients of x, y, z, which enter into the expression of $a_{22} + b_{11} - 2h_{12}$ destroy each other, and this combination contains really only second differential coefficients of x, y, z.

Similarly

$$f_{31} = \quad 31.23 + 33.12 + \quad 3.123 \qquad + 2.133,$$

$$g_{23} = \quad 31.23 + 33.12 + \quad 3.123 + 1.233$$

$$c_{12} = 2.31.23 \qquad + 2.3.123$$

$$h_{33} = 2.31.23 \qquad\qquad + 1.233 + 2.133,$$

and thence

$$f_{31} + g_{23} - c_{12} - h_{33} = -2\,(31.23 - 33.12),$$

so that here again we have a combination containing only second differential coefficients of x, y, z.

There are thus, in all, the six combinations

$$b_{33} + c_{22} - 2f_{23} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{A}),$$

$$c_{11} + a_{33} - 2g_{31} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{B}),$$

$$a_{22} + b_{11} - 2h_{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{C}),$$

$$g_{12} + h_{31} - \ a_{23} - f_{11} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{F}),$$

$$h_{23} + f_{12} - \ b_{31} - g_{22} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{G}),$$

$$f_{31} + g_{23} - \ c_{12} - h_{33} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{H}),$$

each really containing only second differential coefficients of x, y, z; and we thus understand how each of these combinations may be expressible in terms of the first differential coefficients of (a, b, c, f, g, h). We have, in fact, for

thus expressing these combinations the six equations called (𝔄), (𝔅), (ℭ), (𝔉), (𝔊), (ℌ) about to be obtained, and which are the generalisations of Warren's six equations for normal coordinates.

I consider the several determinants of the form

$$\begin{vmatrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

in all 18, since the suffixes for the top row may be 11, 22, 33, 23, 31, 12, and those for the second and third rows 2, 3; 3, 1; or 1, 2. Each determinant is a linear function of second differential coefficients of x, y, z (thus the determinant written down is $= \xi_3 x_{11} + \eta_3 y_{11} + \zeta_3 z_{11}$), and as such it can, by what precedes, be expressed by means of the first differential coefficients of the a, b, c, f, g, h. Thus, if the determinant above written down be called $\square$, writing ξ_3, η_3, ζ_3 for α, β, γ, we have

$$\begin{vmatrix} \xi_3, & \eta_3, & \zeta_3, & \square \\ x_1, & y_1, & z_1, & \frac{1}{2}a_1 \\ x_2, & y_2, & z_2, & h_1 - \frac{1}{2}a_2 \\ x_3, & y_3, & z_3, & z_1 - \frac{1}{2}a_3 \end{vmatrix} = 0,$$

that is

$$L.\square = - \begin{vmatrix} \xi_3, & \eta_3, & \zeta_3 \\ \frac{1}{2}a_1, & x_1, & y_1, & z_1 \\ h_1 - \frac{1}{2}a_2, & x_2, & y_2, & z_2 \\ g_1 - \frac{1}{2}a_3, & x_3, & y_3, & z_3 \end{vmatrix}$$

where the right-hand side is

$$\begin{aligned} = & \quad \tfrac{1}{2}a_1 . \xi_1\xi_3 + \eta_1\eta_3 + \zeta_1\zeta_3 \\ & - (h_1 - \tfrac{1}{2}a_2) . - (\xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3) \\ & + (g_1 - \tfrac{1}{2}a_3) . \xi_3^2 + \eta_3^2 + \zeta_3^2, \end{aligned}$$

which is $= G . \frac{1}{2}a_1 + F (h_1 - \frac{1}{2}a_2) + C (g_1 - \frac{1}{2}a_3)$,

or as this might be written

$$= (G, F, C \mathcal{X} a_1, h_1, g_1) - \tfrac{1}{2} (G, F, C \mathcal{X} a_1, a_2, a_3),$$

but retaining the former form, the result is

$$L.\square = G . \tfrac{1}{2}a_1 + F (h_1 - \tfrac{1}{2}a_2) + C (g_1 - \tfrac{1}{2}a_3),$$

and it is now very easy to write down the complete system of the 18 equations; viz. if the determinant above written down be called 11.1.2, and so in other cases, then we have

	2.3	A	H	G
	3.1	H	B	F
	1.2	G	F	C
L . 11	„ =	$\frac{1}{2}a_1$	$h_1 - \frac{1}{2}a_2$	$g_1 - \frac{1}{2}a_3$
„ 22	„ =	$h_2 - \frac{1}{2}b_1$	$\frac{1}{2}b_2$	$f_2 - \frac{1}{2}b_3$
„ 33	„ =	$g_3 - \frac{1}{2}c_1$	$f_3 - \frac{1}{2}c_2$	$\frac{1}{2}c_3$
„ 23	„ =	$\frac{1}{2}(-f_1 + g_2 + h_3)$	$\frac{1}{2}b_3$	$\frac{1}{2}c_2$
„ 31	„ =	$\frac{1}{2}a_3$	$\frac{1}{2}(f_1 - g_2 + h_3)$	$\frac{1}{2}c_1$
„ 12	„ =	$\frac{1}{2}a_2$	$\frac{1}{2}b_1$	$\frac{1}{2}(f_1 + g_2 - h_3)$

read for instance

$$L.11.1.2 = G.\tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a) + C(g_1 - \tfrac{1}{2}a_3),$$

the equation obtained above.

There are eighteen functions as shown by the following diagram:

	2.3	3.1	1.2
$(22)(33) - (23)^2$	2.3		
$(33)(11) - (31)^2$	3.1		
$(11)(22) - (12)^2$	1.2		
$(31)(12) - (11)(23)$	2.3		
$(12)(23) - (22)(31)$	3.1		
$(23)(31) - (33)(12)$	1.2,		

viz. in any line of the diagram the bracketed duads may belong to any one at pleasure, but all to the same, of the three pairs 2.3, 3.1, 1.2; thus the first line might be

$$(22.3.1)(33.3.1) - (23.3.1)^2,$$

or, instead of the 3.1, we might have 2.3 or 1.2. But of the 18 functions I distinguish 6, viz. those in which for the six lines respectively the pairs are 2.3, 3.1, 1.2, 2.3, 3.1, 1.2, as shown in the diagram. Each of these six functions can be obtained under two different forms, and by equating these we have the equations (𝔄), (𝔅), (ℭ), (𝔇), (𝔈), (𝔉), before

referred to; thus ($\mathfrak{C}$) is obtained by equating two different forms of the function $(11.1.2)(22.1.2)-(12\ 1.2)^2$; and ($\mathfrak{H}$) is obtained by equating two different forms of the function $(23.1.2)(31.1.2)-(33.1.2)(12.1.2)$.

[The determinants 11.1.2, &c., may be denoted by accented letters a, b, c, f, g, h, as follows:

$$\left|\begin{matrix} x_{11}, & x_{22}, & x_{33}, & x_{23}, & x_{31}, & x_{12} \\ y_{11}, & y_{22}, & y_{33}, & y_{23}, & y_{31}, & y_{12} \\ z_{11}, & z_{22}, & z_{33}, & z_{23}, & z_{31}, & z_{12} \end{matrix}\right| \left|\begin{matrix} x_2, & x_3 \\ y_2, & y_3 \\ z_2, & z_3 \end{matrix}\right| = a', b', c', f', g', h',$$

$$\left|\begin{matrix} \quad\quad \text{"} \quad\quad & \quad\quad \text{"} \quad\quad \end{matrix}\right| \left|\begin{matrix} x_3, & x_1 \\ y_3, & y_1 \\ z_3, & z_1 \end{matrix}\right| = a'', b'', c'', f'', g'', h'',$$

$$\left|\begin{matrix} \quad\quad \text{"} \quad\quad & \quad\quad \text{"} \quad\quad \end{matrix}\right| \left|\begin{matrix} x_1, & x_2 \\ y_1, & y_2 \\ z_1, & z_2 \end{matrix}\right| = a''', b''', c''', f''', g''', h''',$$

viz.

$$\left|\begin{matrix} x_{11}, & x_2, & x_3 \\ y_{11}, & y_2, & y_3 \\ z_{11}, & z_2, & z_3 \end{matrix}\right| = a' \text{ &c.}$$

In this notation ($\mathfrak{C}$) is obtained by equating two values of $a'''b''' - (h''')^3$, and $\mathfrak{H}$ by equating two values of $f'''g''' - c'''h'''$].

The forms which I call the second are those given by the immediate substitution of the foregoing values of (11.12), &c.; for obtaining the first forms I proceed to calculate those of ($\mathfrak{C}$) and ($\mathfrak{H}$).

Forming by the ordinary rule the product of the determinants (11.1.2) and (22.1.2), which are

$$\left|\begin{matrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right| \text{ and } \left|\begin{matrix} x_{22}, & y_{22}, & z_{22} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right|,$$

this is

$$\left|\begin{matrix} 11.22, & 1.11, & 2.11 \\ 1.22, & 1.1, & 1.2 \\ 2.22, & 1.2, & 2.2 \end{matrix}\right|,$$

where 11.22 denotes $x_{11}x_{22} + y_{11}y_{22} + z_{11}z_{22}$, and the like for the other symbols; in like manner the square of the determinant (12.1.2), that is of

$$\left|\begin{matrix} x_{12}, & y_{12}, & z_{12} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right|,$$

is $$\begin{vmatrix} 12.12, & 1.12, & 2.12 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 2.1, & 2.2 \end{vmatrix},$$

or, observing that in the two resulting determinants the terms 11.22 and 12.12 are multiplied by the same factor, the expression for the difference gives

$$(11.1.2)(22.1.2) - (12.1.2)^2$$

$$= (11.22 - 12.12) \begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix} + \begin{vmatrix} 0, & 1.11, & 2.11 \\ 1.22, & 1.1, & 1.2 \\ 2.22, & 1.2, & 2.2 \end{vmatrix}$$

$$- \begin{vmatrix} 0, & 1.12, & 2.12 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 1.2, & 2.2 \end{vmatrix},$$

containing $11.22 - 12.12$, which by what precedes is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12});$$

the other terms are also known, viz. the whole value is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12})(ab - h^2)$$

$$+ \begin{vmatrix} 0, & \tfrac{1}{2}a, & h_1 - \tfrac{1}{2}a_2 \\ h_2 - \tfrac{1}{2}b_1, & a, & h \\ \tfrac{1}{2}b_2, & h, & b \end{vmatrix} - \begin{vmatrix} 0, & \tfrac{1}{2}a_2, & \tfrac{1}{2}b_1 \\ \tfrac{1}{2}a_2, & a, & h \\ \tfrac{1}{2}b_1, & h, & b \end{vmatrix},$$

which is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12})(ab - h^2)$$
$$+ a(\tfrac{1}{4}b_1^2 - \tfrac{1}{2}b_2h_1 + \tfrac{1}{4}a_2b_2)$$
$$+ b(\tfrac{1}{4}a_2^2 - \tfrac{1}{2}a_1h_2 + \tfrac{1}{4}a_1b_1)$$
$$+ h(\tfrac{1}{4}a_1b_2 - \tfrac{1}{4}a_2b_1 + h_1h_2 - \tfrac{1}{2}a_2h_2 - \tfrac{1}{2}b_1h_1)$$
$$= (ab - h^2)k, \text{ where } k \text{ is the measure of curvature.}$$

In the same way we have

$$(23.1.2)(31.1.2) - (33.1.2)(12.1.2)$$

$$= \begin{vmatrix} 23.31, & 1.31, & 2.31 \\ 1.23, & 1.1, & 1.2 \\ 2.23, & 1.2, & 2.2 \end{vmatrix} - \begin{vmatrix} 12.33, & 1.33, & 2.33 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 1.2, & 2.2 \end{vmatrix},$$

which is

$$= (23.31 - 12.33) \begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix} + \begin{vmatrix} 0\ , & 1.13, & 2.13 \\ 1.23, & 1.1, & 1.2 \\ 2.23, & 1.2, & 2.2 \end{vmatrix} - \begin{vmatrix} 0\ , & 1.33, & 2.33 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 1.2, & 2.2 \end{vmatrix},$$

containing $23.31 - 12.33$, which by what precedes is

$$= -\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33});$$

the other terms are also known, and the value is

$$= -\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33})(ab - h^2)$$

$$+ \begin{vmatrix} 0 & , \tfrac{1}{2}a_3, & \tfrac{1}{2}(f_1 - g_2 + h_3) \\ \tfrac{1}{2}(-f_1 + g_2 + h_3), & a\ , & h \\ \tfrac{1}{2}b_3 & , h\ , & b \end{vmatrix} - \begin{vmatrix} 0\ , & g_3 - \tfrac{1}{2}c_1, & f_3 - \tfrac{1}{2}c_2 \\ \tfrac{1}{2}a_2, & a\ , & h \\ \tfrac{1}{2}b_1, & h\ , & b \end{vmatrix},$$

or finally this is

$$= -\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33})(ab - h^2)$$

$$+ a(\tfrac{1}{2}b_1 f_3 - \tfrac{1}{4}b_3 f_1 - \tfrac{1}{4}b_1 c_2 + \tfrac{1}{4}b_3 g_2 - \tfrac{1}{4}b\, h_3)$$

$$+ b(\tfrac{1}{2}a_2 g_2 - \tfrac{1}{4}a_3 g_3 - \tfrac{1}{4}a_2 c_1 + \tfrac{1}{4}a_3 f_1 - \tfrac{1}{4}a_3 h_3)$$

$$+ h\{\tfrac{1}{4}a_3 b_3 + \tfrac{1}{4}b_1 c_1 + \tfrac{1}{4}a_2 c_2 - \tfrac{1}{2}b_1 g_3 - \tfrac{1}{2}a_2 f_3 + \tfrac{1}{4}h_3^2 - \tfrac{1}{4}(f_1 - g_2)^2\}.$$

The remaining four of the six functions can of course be obtained from the two just found by a cyclical interchange of the letters and suffix-numbers 1, 2, 3, and it is not worth while to write down the values.

The two values of $(11.1.2)(22.1.2) - (12.1.2)^2$ were

$$-(a_{22} + b_{11} - 2h_{12})(ab - h^2) + a(\tfrac{1}{4}b_1^2 - \tfrac{1}{2}b_2 h_1 + \tfrac{1}{4}a_2 b_2)$$
$$+ b(\tfrac{1}{4}a_2^2 - \tfrac{1}{2}a_1 h_2 + \tfrac{1}{4}a_1 b_1)$$
$$+ h\{\tfrac{1}{4}(a_1 b_2 - a_2 b_1) + h_1 h_2 - \tfrac{1}{2}b_1 h_1 - \tfrac{1}{2}a_2 h_2),$$

and

$$\{G.\tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3)\}.\{G(h_2 - \tfrac{1}{2}b_1) + F.\tfrac{1}{2}b_2 + C(f_2 - \tfrac{1}{2}b_3)\}$$
$$- \{G.\tfrac{1}{2}a_2 + F.\tfrac{1}{2}b_1 + C.\tfrac{1}{2}(f_1 + g_2 + h_3)\}^2,$$

where, in the first value, for a, b, h, $ab - h^2$ we must write $L^{-2}(BC - F^2)$, $L^{-2}(CA - G^2)$, $L^{-2}(FG - CH)$, and L^2C; making this change, multiplying by 4 and equating, we obtain

$$\begin{aligned}
&- 2L^2C\,(a_{22} + b_{11} - 2h_{12}) \\
&+ (BC - F^2)\,(b_1^2 - 2b_2h_1 + a_2b_2) \\
&+ (CA - G^2)\,(a_2^2 - 2a_1h_2 + a_1b_1) \\
&+ (FG - CH)\,(a_1b_2 - a_2b_1 + 4h_1h_2 - 2b_1h_1 - 2a_2h_2) \\
&- \{Ga_1 + F(2h_1 - a_2) + C(2g_1 - a_3)\}\{G(2h_2 - b_1) + Fb_2 + C(2f_2 - b_3)\} \\
&+ \{Ga_2 + Fb_1 + C(f_1 + g_2 - h_3)\}^2 \qquad = 0.
\end{aligned}$$

Developing the fourth and fifth lines, it appears that in this expression the coefficients of F^2, G^2 and FG each of them vanish; the whole equation is thus divisible by C, and omitting this factor throughout, the equation becomes $0 =$

$$\begin{aligned}
&- 2L^2\,(a_{22} + b_{11} - 2h_{12}) \\
&+ A\,(a_2^2 - 2ah_{12} + a_1b_1) \\
&+ B\,(b_1^2 - 2b_2h_1 + a_2b_2) \\
&+ C\{(-a_3b_3 + 2a_3f_2 + 2b_3g_1 - 4f_2g_1 + (f_1 + g_2 + h)^2\} \\
&+ F\{(a_3b_2 - a_2b_3) + 2(b_1g_2 - b_2g_1) + 2(b_3h_1 - b_1h_3) + 2(a_2f_2 + b_1f_1 - 2h_1f_2)\} \\
&+ G\{a_1b_3 - a_3b_1 + 2(a_2f_1 - a_1f_2) + 2(a_3h_2 - a_2h_3) + 2(a_2g_2 + b_1g_1 - 2g_1h_2)\} \\
&+ H\{\qquad\qquad 2\,(a_2h_2 + b_1h_1 - 2h_1h_2)\}\,;
\end{aligned}$$

this is substantially the required equation (Ω), but the form of it may be greatly simplified.

Forming the identity, $0 =$

$$\begin{aligned}
2L\,[(a_2 - h_1)\,L_2 + (b_1 - h_2)\,L_1] = \\
&- A\,(a_1b_1 - a_1h_2 - a_2h_1 + a_2^2) \\
&- B\,(b_1^2 - b_1h_2 - b_2h_1 + a_2b_2) \\
&- C\,(b_1c_1 + a_2c_2 - c_1h_2 - c_2h_1) \\
&- 2F\,(b_1f_1 + a_2f_2 - f_1h_2 - f_2h_1) \\
&- 2G\,(b_1g_1 + a_2g_2 - g_1h_2 - g_2h_1) \\
&- 2H\,(a_2h_2 + b_1h_1 - 2h_1h_2),
\end{aligned}$$

we add hereto the last preceding equation; the coefficients of A, B, F, G, H thus assume new and simple forms, but the coefficient of C requires a further transformation.

Assume

$$\Omega = (b_1c_1 - f_1^2) + (c_2a_2 - g_2) + (a_3b_3 - h_3^2)$$
$$+ (g_2h_3 + g_3h_2 - a_2f_3 - a_3f_2) + (h_3f_1 + h_1f_3 - b_1g_3 - b_3g_1)$$
$$+ (f_1g_2 + f_2g_1 - c_1h_2 - c_2h_1),$$

then if we add to the equation $-C\Omega + C$ into this value, the coefficient of C takes its proper form, and the equation is

$$-2L^2(a_{22} + b_{11} - 2h_{12})$$
$$+2L[(b_1 - h_2)L_1 + (a_2 - h_1)L_2] - C\Omega$$
$$+A\{-(ah_2 - a_2h)\}$$
$$+B\{\quad (b_1h_2 - b_2h_1)\}$$
$$+C\{-(a_2f_3 - a_3f_2) + (b_3g_1 - b_1g_3) - (g_2h_3 - g_3h_2) - (h_3f_1 - h_1f_3)$$
$$+3(f_1g_2 - f_2g_1)\}$$
$$+F\{-(a_2b_3 - a_3b_2) + 2(b_1g_2 - b_2g_1) + 2(b_3h_1 - b_1h_3) - 2(h_1f_2 - h_2f_1)\}$$
$$+G\{-(a_3b_1 - a_1b_3) - 2(a_1f_2 - a_2f_1) - 2(a_2h_3 - a_3h_1) - 2(g_1h_2 - g_2h_1)\}$$
$$+H\{-(a_1b_2 - a_2b_1)\}$$
$$= 0,$$

where, as a verification, observe that the letters (a, b), (f, g) may be interchanged if at the same time we interchange the suffixes 1 and 2.

The two values of $(23.1.2)(31.1.2) - (33.1.2)(12.1.2)$ are

$$-\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33})(ab - h^2)$$
$$+a(\tfrac{1}{2}b_1f_3 - \tfrac{1}{4}b_3f_1 - \tfrac{1}{4}b_1c_2 + \tfrac{1}{4}b_3g_2 - \tfrac{1}{4}b_3h_3)$$
$$+b(\tfrac{1}{2}a_2g_3 - \tfrac{1}{4}a_3g_2 - \tfrac{1}{4}a_2c_1 + \tfrac{1}{4}a_3f_1 - \tfrac{1}{4}a_3h_3)$$
$$+h\{\tfrac{1}{4}a_3h_3 + \tfrac{1}{4}b_1c_1 + \tfrac{1}{4}a_2c_2 - \tfrac{1}{2}b_1g_3 - \tfrac{1}{2}a_2f_3 + \tfrac{1}{4}h_3^2 - \tfrac{1}{4}(f_1 - g_2)^2\},$$

and

$$\{G.\tfrac{1}{2}(-f_1+g_2+h_3)+F.\tfrac{1}{2}b_3+C.\tfrac{1}{2}c_2\}\{G.\tfrac{1}{2}a_3+F.\tfrac{1}{2}(f_1-g_2+h_3)+C.\tfrac{1}{2}c_1\}$$
$$-\{G(g_3-\tfrac{1}{2}c_1)+F(f_3-\tfrac{1}{2}c_2)+C.\tfrac{1}{2}c_3\}\{G.\tfrac{1}{2}a_2+F.\tfrac{1}{2}b_1+C.\tfrac{1}{2}(f_1+g_2-h_3)\},$$

and here for a, b, h, $ab-h^2$, substituting their values, multiplying by 4, and equating, we have

$$-2L^2C\,(g_{23}+f_{31}-c_{12}-h_{33})$$
$$+(BC-F^2)\,(2b_1f_3-b_3f_1-b_1c_2+b_3g_2-b_3h_3)$$
$$+(CA-G^2)\,(2a_2g_3-a_3g_2-a_2c_1+a_3f_1-a_3h_3)$$
$$+(FG-CH)(a_3b_3+b_1c_1+a_2c_2-2b_1g_3-2a_2f_3+h_3^2-f_1^2+2f_1g_2-g^2)$$
$$-\{G(-f_1+g_2+h_3)+Fb_3+Cc_2\}\{Ga_3+F(f_1-g_2+h_3)+Cc_1\}$$
$$+\{G(2g_3-c_1)+F(2f_3-c_2)+Cc_3\}\{(Ga_2+Fb_1+C(f_1+g_2-h_3)\}=0.$$

Here again the terms in F^2, FG, G^2 all vanish, hence the whole equation divides by C, and throwing this factor out we have

$$-2L^2\,(g_{23}+f_{31}-c_{12}-h_{33})$$
$$+A\,(\;2a_2g_3-a_3g_2-\;a_2c_1+\;a_3f_1-\;a_3h_3)$$
$$+B\,(\;2b_1f_3-b_3f_1-\;b_1c_2+\;b_3g_2-\;b_3h_3)$$
$$+C\,(-\;c_1c_2+c_3f_1+\;c_3g_2-\;c_3h_3)$$
$$+F\,\{-\;(b_3c_1-b_1c_3)-2c_2f_1+2f_1f_3+2g_2f_3-2f_3h_3\}$$
$$+G\,(\;\;a_2c_3-a_3c_2-2c_1g_2+2f_1g_3+2g_2g_3-2g_3h_3)$$
$$+H\,(-\;a_3b_3-b_1c_1-\;a_2c_2+2b_1g_3+2a_2f_3-\;h_3^2+f_1^2-2f_1g_2+g_2^2)$$
$$=0,$$

which is substantially the required equation (*H*); but the form has to be altered; first multiply by 2, then forming the expression

$$2L\,[(f_3-c_2)\,L_1+(g_3-c_2)\,L_2+(f_1+g_2-2h_3)\,L_3]$$
$$=\quad(f_3-\;c_2)\,(Aa_1+Bb_1+Cc_1+2Ff_1+2Gg_1+2Hh_1)$$
$$+(g_3-\;c_1)\,(Aa_2+Bb_2+Cc_2+2Ff_2+2Gg_2+2Hb_2)$$
$$+(f_1+g_2-2h_3)\,(Aa_3+Bb_3+Cc_3+2Ff_3+2Gg_3+2Hb_3)$$

this is

$$
\begin{aligned}
= \quad & A(- a_1c_2 - a_2c_1 + a_1f_3 + a_3f_1 + a_2g_3 + a_3g_2 - 2a_3h_3) \\
+ \quad & B(- b_1c_2 - b_2c_1 + b_1f_3 + b_3f_1 + b_2g_3 + b_3g_2 - 2b_3h_3) \\
+ \quad & C(-2c_1c_2 \qquad + c_1f_3 + c_3f_1 + c_2g_3 + c_3g_2 - 2c_3h_3) \\
+ \quad & 2F(- c_1f_2 - c_2f_1 + 2f_1f_3 + \qquad f_2g_3 + f_3g_2 - 2f_3h_3) \\
+ \quad & 2G(- c_1g_2 - c_2g_1 + f_1g_3 + f_3g_1 + 2g_2g_3 - 2g_3h_3) \\
+ \quad & 2H(- c_1h_2 - c_2h_1 + g_2h_3 + g_3h_2 + h_3f_1 + h_1f_3 - 2h_3^2),
\end{aligned}
$$

and then adding this expression

$$(f_3 - c_2)\,L_1 + (g_3 - c_1)\,L_2 + (f_1 + g_2 - 2h_3)\,L_3$$

— its foregoing value, we have an equation with new coefficients of A, B, C, F, G, H, all of which, except that of H, are in the proper form, and for the coefficient of H, recurring to the foregoing value of Ω, we must add to the equation $-2H\Omega + 2H$ into value of Ω. The final result is

$$
\begin{aligned}
& - 4L^2(g_{23} + f_{31} - c_{12} - h_{33}) \\
& + 2L[(f_3 - c_2)\,L_1 + (g_3 - c_1)\,L_2 + (f_1 + g_2 - 2h_3)\,L_3] - 2H\Omega \\
& \quad + A\{\ 3(a_2g_3 - a_3g_2) - (c_1a_2 - c_2a_1) + (a_3f_1 - a_1f_3)\} \\
& \quad + B\{-3(b_3f_1 - b_1f_3) - (b_1c_2 - b_2c_1) - (b_2g_3 - b_3g_2)\} \\
& \quad + C\{\ (c_3f_1 - c_1f_3) - (c_2g_3 - c_3g^2) \qquad\qquad\ \} \\
& \quad + 2F\{-\ (b_3c_1 - b_1c_3) + (c_1f_2 - c_2f_1) - (f_2g_3 - f_3g_2)\} \\
& \quad + 2G\{-\ (c_2a_3 - c_3a_2) - (c_1g_2 - c_2g_1) - (f_3g_1 - fg_3)\} \\
& \quad + 2H\{-\ (b_3g_1 - b_1g_3) + (a_2f_3 - a_3f_2) \qquad\qquad\ \} = 0,
\end{aligned}
$$

which, it will be observed, remains unaltered by the interchange of a and b, f and g, A and B, F and G, the suffix numbers 1 and 2 being at the same time interchanged.

We have thus the required six equations, in which

$$L^2 = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

$$
\begin{aligned}
\Omega = {} & b_1c_1 - f_1^2 + c_2a_2 - g_2^2 + a_3b_3 - h_3^2 \\
& + (g_2h_3 + g_3h_2 - a_2f_3 - a_3f_2) + (b_3f_1 + h_1f_3 - b_3g_1 - b_1g_3) \\
& \qquad + (f_1g_2 + f_2g_1 - a_1f_2 - a_2f_1),
\end{aligned}
$$

and where I write also, for shortness, ab12 for $a_1b_2 - a_2b_1$, &c. viz. the equations are

$$0 = -2L^2(b_{33} + c_{22} - 2f_{23}) \qquad (\mathfrak{A}).$$

$$+2L[\ .\ (c_2 - f_3)L_2 + (b_3 - f_2)L_3] - A\Omega$$

$$+A\{\ .\ - bg31 + ch12 + 3.gh23 - hf31 - fg12\}$$

$$+B\{- bf23 \}$$

$$+C\{+ cf23 \}$$

$$+F\{- bc23 \}$$

$$+G\{- bc31 + 2.cf12\ .\ +2.ch23\ .\ -2.fg23\}$$

$$+H\{- bc12 - 2.bf31 - 2.bg23\ .\ .\ -2.hf23\}$$

$$0 = -2L^2(c_{11} + a_{33} - 2g_{31}) \qquad (\mathfrak{B}).$$

$$+2L[(c_1 - g_3)L_1\ .\ +(a_3 - g_1)L_3] - B\Omega$$

$$+A\{\ .\ ag31 \}$$

$$+B\{\ af23\ .\ - ch12 - gh23 + 3.hf31 - fg31\}$$

$$+C\{\ -cg31 \}$$

$$+F\{- ca23\ .\ -2.cg12 - 2.ch31 - 2.fg31\}$$

$$+G\{- ca31 \}$$

$$+H\{- ca12 + 2.af31 + 2.ag23\ 2.gh31\}$$

$$0 = -2L^2(a_{22} + b_{11} - 2h_{12}) \qquad (\mathfrak{C}).$$

$$+2L[(b_1 - h_2)L_1 + (a_2 - h_1)L_2\ .\ ] - C\Omega$$

$$+A\{\ .\ .\ - ah12 \}$$

$$+B\{\ .\ .\ + bh12 \}$$

$$+C\{- af23 + bg31\ .\ - gh23 - hf31 + 3.fg12\}$$

$$+F\{- ab23\ .\ .\ +2.bg12 + 2.bh31 - 2.hf12\}$$

$$+G\{- ab31\ .\ -2.af12\ .\ +2.ah23 - 2.gh12\}$$

$$+H\{- ab12 \}$$

$$0 = -4L^2(g_{12} + h_{31} - a_{23} - f_{11}) \qquad (\mathfrak{F}).$$

$$+2L[(g_2 + h_3 - 2f_1)L_1 + (g_1 - a_3)L_2 + (h_1 - a_2)L_3] \; -2F\Omega$$

$$+ A\{\quad . \quad ag12 - \quad ah31 \quad\}$$

$$+ B\{-ab23 + \quad bg12 + 3.bh31 \quad\}$$

$$+ C\{-ca23 - 3.cg12 - \quad ch31 \quad\}$$

$$+2F\{\quad . \quad + \quad bg31 - \quad ch12 \quad\}$$

$$+2G\{-ca12 + \quad ag23 \quad . \quad - gh31\}$$

$$+2H\{-ab31 \quad . \quad - \quad ah23 - gh12\}$$

$$0 = -4L^2(h_{23} + f_{12} - b_{31} - g_{22}) \qquad (\mathfrak{G}).$$

$$+2L[(f_2 - b_3)L_1 + (h_3 + f_1 - 2g_2)L_2 + (h_2 - b_1)L_3] \; -2G\Omega$$

$$+ A\{-ab31 - 3.ah23 - \quad af12 \quad\}$$

$$+ B\{\quad . \quad bh23 - \quad bf12 \quad\}$$

$$+ C\{-bc31 + \quad ch23 + 3.cf12 \quad\}$$

$$+2F\{-bc12 \quad . \quad - \quad bf31 - hf23\}$$

$$+2G\{\quad . \quad + \quad ch12 - \quad af23 \quad . \quad\}$$

$$+2H\{-ab23 + \quad bh31 \quad . \quad - hf12\}$$

$$0 = -4L^2(f_{31} + g_{23} - c_{12} - h_{33}) \qquad (\mathfrak{H}).$$

$$+2L[(f_3 - c_2)L_1 + (g_3 - c_1)L_2 + (f_1 + g_2 - 2h_3)L_3] \; -2H\Omega$$

$$+ A\{-ca12 + \quad af31 + 3.ag23 \quad\}$$

$$+ B\{-bc12 - 3.bf31 - \quad bg23 \quad\}$$

$$+ C\{\quad . \quad cf31 - \quad cg23 \quad\}$$

$$+ F\{-bc31 + \quad cf12 \quad . \quad - fg23\}$$

$$+ G\{-ca23 \quad . \quad - \quad cg12 - fg31\}$$

$$+ H\{\quad . \quad + \quad af23 - \quad bg31 \quad . \quad\}$$

It would be possible in these equations to introduce the symbols AB12, &c., in place of ab12, &c., and then writing $A = B = C = 1$, all these symbols other than those where the letters are GH, HF or FG would vanish, and we should obtain Mr. Warren's six equations for normal coordinates. But in the general case it would seem that there is not any advantage in the introduction of the new symbols AB12, &c., and I retain by preference the equations in the form in which I have given them

To the foregoing may be joined a symmetrical equation obtained (as by Mr Warren) by multiplying the several equations by a, b, c, f, g, h respectively, and adding; the result is in the first instance obtained in the form

$$-2L^2\Theta + 2L\Psi - 3L^2\Omega + \square = 0 \ldots\ldots\ldots (\mathfrak{M}),$$

where

$$\begin{aligned}\Theta = \quad & a\,(b_{33} + c_{22} - 2f_{23}) \\ & + \; b\,(c_{11} + a_{33} - 2g_{31}) \\ & + \; c\,(a_{22} + b_{11} - 2h_{12}) \\ & + 2f\,(g_{12} + h_{31} - a_{23} - f_{11}) \\ & + 2g\,(h_{23} + f_{12} - b_{31} - g_{22}) \\ & + 2h\,(f_{31} + g_{23} - c_{12} - h_{25}).\end{aligned}$$

For Ψ, collecting the terms which contain L_1, L_2, L_3 respectively, and attending to the values of (A, B, C, F, G, H),

$$= (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$

this is easily reduced to the form

$$\Psi = (A_1 + H_2 + G_3)\,L_1 + (H_1 + B_2 + F_3)\,L_2 + (G_1 + F_2 + C_3)\,L_3.$$

The term in Ω is

$$= -(Aa + Bb + Cc + 2Ff + 2Gg + 2Hh)\,\Omega,$$

$$= -3L^2\Omega \text{ as above.}$$

For the calculation of □, collecting the terms, we have □ =

23	A	B	C	F	G	H
bc				− a		
ca			− f	− b	− 2h	
ab		− f		− c		− 2g
af		b − c		− 2g	2h	
bf		− a				
cf			a			
ag	3h			2f	2b	
bg		− h			− 2a	
cg			− h			
ah	− 3g			− 2c	− 2f	
bh		g				
ch			g		2a	
gh	− 3a	− b	− c			
hf				− 2g	− 2a	
fg				− 2h	− 2a	

31	A	B	C	F	G	H
bc				− g − 2h − a		
ca					− b	
ab		− g				− c − 2f
af	h					2b
bf		− 3h		− 2g	− 2a	
cf				h		
ag	b					
bg		− a	+ c	2f	− 2h	
cg				− b		
ah	− f					
bh		3f		2c	2g	
ch				− f − 2b		
gh					− 2f − 2b	
hf		− a	3b − c			
fg				− 2b − 2h		

12	A	B	C	F	G	H
bc		− h		− 2g		− a
ca	− h				− 2f − b	
ab						− c
af	− g				− 2c	
bf		− g				
cf				3g	2h	2a
ag	f					
bg		f		2c		
cg				− 3f − 2b − 2h		
ah	− c					
bh		c				
ch		a − b		− 2f + 2g		
gh					− 2c − 2f	
hf				− 2c	− 2g	
fg				− a − b + 3c		

viz. this is

□ = bc23 . − aF + bc31(− g *C* − 2hF − aG) + bc12(− hB − 2gF − aH)

+ &c.

Some however of the coefficients require reduction; for instance that of bc31 is = − (a, h, g)(G, F, C) − hF, = − hF; and so − hB − 2gF − aH is = − (a, h, g)(H, B, F) − gF, = − gF.

After these reductions the value is found to be $\square =$

bc23.−aF	+ bc31.−hF	+ bc12.−gF
+ ca23.−hG	+ ca31.−bG	+ ca12.−fG
+ ab23.−gH	+ ab31.−fH	+ ab12.−cH
+ af23(−gG+hH)	+ af31(bH−fG)	+ af12(fH−cG)
+ bf23.−aB	+ bf31.−hB	+ bf12.−gB
+ cf23.aC	+ cf31.hC	+ cf12.gC
+ ag23.hA	+ ag31.bA	+ ag12.fA
+ bg23(gF−aH)	+ bg31(−hH+fF)	+ bg12(cF−gH)
+ cg23.−hC	+ cg31.−bC	+ cg12.−fC
+ ah23.−gA	+ ah31.−fA	+ ah12.−cA
+ bh23.gB	+ bh31.fB	+ bh12.cB
+ ch23(aG−hF)	+ ch31.(hG−bF)	+ ch12(−fF+gG)
+ gh23(3aA−bB−cC	+ gh31.2hA	+ gh12.2gA
+ hf23.2hB	+ hf31(−aA+3bB−cC)	+ hf12.2fB
+ fg23.2gC	+ fg31.2fC	+ fg12(−aA−bB+3cC);

or, arranging the terms in a different order, this may be written

$\square =$

A{ a(3.gh23-hf31-fg12) +b.ag31-c.ah12 -c.ah12
+f(ag12-ah31) +g(2.gh12-ah23) +h(2.gh31+ag23)}

+B{-a.bf23 +b(-gh23+3.hf31-fg12) +c.bh12
+f(2.hf12+bh31) +g(bh23-bf12) +h(2.hf23-bf31)}

+C{ a.cf23 -b.cg31 +c(-gh23-hf31+3.fg12)
+f(2.fg31-cg12) +g(2.fg23+cf12) +h(cf31-cg23)}

+F{-a.bc23 -b.ch31 +c.bg12
+f(bg31-ch12) +g(bg23-bc12) -h(ch23 + bc31)}

+G{+a.ch23 -b.ca31 -c.af12
-f(af31+ca12) +g(ch12-af23) +h(ch31-ca23)}

+H{-a.bg23 +b.af31 -c.ab12
+f(af12-ab31) -g(bg12+ab23) +h(af23-bg31)}.

Attending to the values of A, B, C, F, G, H, we have

$$\begin{aligned}
A_{11} + B_{22} + C_{33} + 2F_{23} + 2G_{31} + 2H_{12} \\
= \quad & bc_{11} + cb_{11} - 2ff_{11} \quad + 2(b_1c_1 - f_1^2) \\
+ \quad & ca_{22} + ac_{22} - 2gg_{22} \quad + 2(c_2a_2 - g_2^2) \\
+ \quad & ab_{33} + ba_{33} - 2hh_{33} \quad + 2(a_3b_3 - h_3^2) \\
& + 2\{gh_{23} + hg_{23} - af_{23} - fa_{23} + g_2h_3 + g_3h_2 - a_2f_3 - a_3f_2\} \\
& + 2\{hf_{31} + fh_{31} - bg_{31} - gb_{31} + h_3f_1 + h_1f_3 - b_3g_1 - b_1g_3\} \\
& + 2\{fg_{12} + gf_{12} - ch_{12} - hc_{12} + f_1g_2 + f_2g_1 - c_1h_2 - c_2h_1\}
\end{aligned}$$

which is in fact $= 2\Theta + 2\Omega$.

Hence, the foregoing equation ($\mathfrak{M}$) may also be written

$$-L^2(A_{11} + B_{22} + C_{33} + 2F_{23} + 2G_{31} + 2H_{12}) + 2L\Psi - L^2\Omega + \square = 0,$$

where Ψ, Ω, $\square$ have their beforementioned values.

In the particular case where $f = 0$, $g = 0$, $h = 0$, we have

$$A, B, C, F, G, H = bc, ca, ab, 0, 0, 0; \quad L^2 = abc;$$

$$\Omega = b_1c_1 + c_2a_2 + a_3b_3;$$

the equation ($\mathfrak{A}$) becomes

$$\begin{aligned}
& -2abc(b_{33} + c_{22}) \\
& \quad + [c_2(a_2bc + b_2ca + c_2ab) + b_3(a_3bc + b_3ca + c_3ab)] \\
& \qquad - bc(b_1c_1 + c_2a_2 + a_3b_3) = 0,
\end{aligned}$$

that is

$$-2abc(b_{33} + c_{22}) - bcb_1c_1 + ca(b_2c_2 + b_3^2) + ab(b_3c_3 + c_2^2) = 0,$$

and the equation ($\mathfrak{F}$) becomes

$$\begin{aligned}
& 4abc.a_{23} - [a_3(a_2bc + b_2ca + c_2ab) + a_2(a_3bc + b_3ca + c_3ab)] \\
& \qquad - ca(a_2b_3 - a_3b_2) - ab(c_2a_3 - c_3a_2) = 0,
\end{aligned}$$

that is

$$4abc.a_{23} - 2bc.a_2a_3 - 2ca.a_2b_3 - 2ab.a_3b_2 = 0.$$

Dividing by $-2abc$ and $4abc$ respectively, and completing the system, the six equations are

$$b_{33} + c_{22} - \frac{b_3^2}{2b} - \frac{c_2^2}{2c} + \frac{b_1c_1}{2a} - \frac{b_2c_2}{2b} - \frac{b_3c_3}{2c} = 0 \quad \ldots(\mathfrak{A}),$$

$$c_{11} + a_{33} - \frac{c_1^2}{2c} - \frac{a_3^2}{2a} - \frac{c_1a_1}{2a} + \frac{c_2a_2}{2b} - \frac{c_3a_3}{2c} = 0 \quad \ldots(\mathfrak{B}),$$

$$a_{22} + b_{11} - \frac{a_2^2}{2a} - \frac{b_1^2}{2b} - \frac{a_1b_1}{2a} - \frac{a_2b_2}{2b} + \frac{a_3b_3}{2c} = 0 \quad \ldots(\mathfrak{C}),$$

$$a_{23} \quad - \frac{a_2a_3}{2a} - \frac{a_2b_3}{2b} - \frac{a_3c_2}{2c} \quad = 0 \quad \ldots(\mathfrak{F}),$$

$$b_{31} \quad - \frac{b_1a_3}{2a} - \frac{b_3b_1}{2b} - \frac{b_3c_1}{2c} \quad = 0 \quad \ldots(\mathfrak{G}),$$

$$c_{12} \quad - \frac{c_1a_2}{2a} - \frac{c_2b_1}{2b} - \frac{c_1c_2}{2c} \quad = 0 \quad \ldots(\mathfrak{H}).$$

These are in fact Lamè's equations, *Leçons sur les coordonées curvilignes, etc.*, Paris (1859), pp. 76, 78, viz. the first of the equations (8), p. 76, is

$$\frac{d^2H}{d\rho_1 d\rho_2} = \frac{1}{H_1}\frac{dH}{d\rho_1}\frac{dH_1}{d\rho_2} + \frac{1}{H_2}\frac{dH}{d\rho_2}\frac{dH_2}{d\rho_1},$$

which, in the notation of the present paper, is

$$(\sqrt{a})_{23} = \frac{1}{\sqrt{b}}(\sqrt{a})_2(\sqrt{b})_3 + \frac{1}{\sqrt{c}}(\sqrt{a})_3(\sqrt{c})_2.$$

Here $$(\sqrt{a})_2 = \tfrac{1}{2}\frac{a_2}{\sqrt{a}},$$

$$(\sqrt{a})_{23} = \tfrac{1}{2}\left(\frac{a_2}{\sqrt{a}}\right)_3 = \tfrac{1}{2}\left\{\frac{a_{23}}{\sqrt{a}} - \frac{\frac{1}{2}a_2a_3}{a\sqrt{a}}\right\};$$

and the equation therefore is

$$\frac{a_{23}}{\sqrt{a}} - \frac{\frac{1}{2}a_2a_3}{a\sqrt{a}} = \tfrac{1}{2}\frac{1}{\sqrt{b}}\frac{a_2}{\sqrt{a}}\frac{b_3}{\sqrt{b}} + \tfrac{1}{2}\frac{1}{\sqrt{c}}\frac{a_3}{\sqrt{a}}\frac{c_2}{\sqrt{c}},$$

which, multiplying by $\sqrt{a}$, gives the foregoing equation

$$a_{23} - \frac{a_2a_3}{a} - \frac{a_2b_3}{b} - \frac{a_3c_2}{c} = 0.$$

The first of the equations (9) p. 78 is

$$\frac{d}{d\rho_1}\left(\frac{1}{H_1}\frac{dH}{d\rho_1}\right)+\frac{d}{d\rho}\left(\frac{1}{H}\frac{dH_1}{d\rho}\right)+\frac{1}{H_2^2}\frac{dH}{d\rho_2}\frac{dH_1}{d\rho_2}=0,$$

which, in the notation of the present paper, is

$$\left\{\frac{(\sqrt{a})_2}{\sqrt{b}}\right\}_2+\left\{\frac{(\sqrt{b})_1}{\sqrt{a}}\right\}_1+\frac{1}{c}(\sqrt{a})_3(\sqrt{b})_3=0.$$

This gives first

$$\tfrac{1}{2}\left(\frac{a_2}{\sqrt{a}\sqrt{b}}\right)_2+\tfrac{1}{2}\left(\frac{b_1}{\sqrt{a}\sqrt{b}}\right)_1+\tfrac{1}{4}\frac{a_3b_3}{c\sqrt{a}\sqrt{b}}=0,$$

and then

$$\begin{aligned}&\tfrac{1}{2}\ \frac{a_{22}}{\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{a_2^2}{a\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{a_2b_2}{b\sqrt{a}\sqrt{b}}+\tfrac{1}{4}\frac{a_3b_3}{c\sqrt{a}\sqrt{b}}=0,\\ &+\tfrac{1}{2}\ \frac{b_{11}}{\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{b_1^2}{b\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{a_1b_1}{a\sqrt{a}\sqrt{b}}\end{aligned}$$

which, multiplying by $\sqrt{a}\sqrt{b}$ and reducing, gives the foregoing equation

$$a_{22}+b_{11}-\frac{a_2^2}{a}-\frac{b_1^2}{b}-\frac{a_1b_1}{a}-\frac{a_2b_2}{b}+\frac{a_3b_3}{c}=0.$$

ON CERTAIN FORMULÆ IN ELLIPTIC FUNCTIONS.

By J. W. L. Glaisher.

Introduction, § 1.

§ 1. IN vol. xv. of *Crelle's Journal** Jacobi refers to the algebraical identity

$$(w-x)(y-z)+(w-y)(z-x)+(w-z)(x-y)=0,$$

and points out that it possesses the remarkable property of remaining true if the differences are replaced by their sines, so that

$$\sin(w-x)\sin(y-z)+\sin(w-y)\sin(z-x)+\sin(w-z)\sin(x-y)=0,$$

which, by putting

$$w-x=a,\ \ x-y=u,\ \ y-z=b$$

---

* "Formulæ novæ in theoria transcendentium ellipticarum fundamentales," pp. 199–204, and *Gesammelte Werke*, vol. i. (1881), pp. 335–341.

may be written

$$\sin a \sin b + \sin u \sin (u+a+b) = \sin (u+a) \sin (u+b).$$

Jacobi proceeds to investigate the corresponding formula in Elliptic Functions, and finds that

$$\operatorname{sn} a \operatorname{sn} b + \operatorname{sn} u \operatorname{sn} (u+a+b) - \operatorname{sn} (u+a) \operatorname{sn} (u+b)$$
$$= k^2 \operatorname{sn} a \operatorname{sn} b \operatorname{sn} (u+a) \operatorname{sn} (u+b) \operatorname{sn} (u+a+b),$$

"quæ est formula nova, maximi momenti per totam theoriam functionum ellipticarum."

Introducing again the differences of four quantities, he writes the formula in the form

$$\operatorname{sn} (w-x) \operatorname{sn} (y-z) + \operatorname{sn} (w-y) \operatorname{sn} (z-x) + \operatorname{sn} (w-z) \operatorname{sn} (x-y)$$
$$+ k^2 \operatorname{sn} (w-x) \operatorname{sn} (w-y) \operatorname{sn} (w-z) \operatorname{sn} (y-z) \operatorname{sn} (z-x) \operatorname{sn} (x-y) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and he also gives the following proof of it, which was communicated to him by Richelot.

Putting

$$w-x = \beta+\gamma, \quad w-y = \gamma+\alpha, \quad w-z = \alpha+\beta,$$
$$y-z = \beta-\gamma, \quad z-x = \gamma-\alpha, \quad x-y = \alpha-\beta,$$

we have
$$\operatorname{sn} (w-x) \operatorname{sn} (y-z) = \frac{\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma}{1 - k^2 \operatorname{sn}^2\beta \operatorname{sn}^2\gamma},$$

$$\operatorname{sn} (w-y) \operatorname{sn} (z-x) = \frac{\operatorname{sn}^2\gamma - \operatorname{sn}^2\alpha}{1 - k^2 \operatorname{sn}^2\gamma \operatorname{sn}^2\alpha},$$

$$\operatorname{sn} (w-z) \operatorname{sn} (x-y) = \frac{\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta}{1 - k^2 \operatorname{sn}^2\alpha \operatorname{sn}^2\beta},$$

whence, if $\operatorname{sn}^2\alpha = t, \quad \operatorname{sn}^2\beta = t', \quad \operatorname{sn}^2\gamma = t'',$

the theorem to be proved is

$$\frac{t'-t''}{1-k^2t't''} + \frac{t''-t}{1-k^2t''t} + \frac{t-t'}{1-k^2tt'}$$
$$= \frac{-k^2(t'-t'')(t''-t)(t-t')}{(1-k^2t't'')(1-k^2t''t)(1-k^2tt')},$$

an algebraical identity, which may be easily verified.

The present note chiefly relates to the class of formulæ to which Jacobi's equation (1) belongs.

Derivation of elliptic function formulæ from trigonometrical formulæ, §§ 2–9.

§ 2. The expression for the tangent of the difference of two quantities in trigonometry bears a close resemblance to to the expression for the product of the sn's of the sum and difference of two quantities, viz. we have

$$\tan(A-B)=\frac{\tan A-\tan B}{1-\tan A\tan B},$$

$$\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta)=\frac{\operatorname{sn}^2\alpha-\operatorname{sn}^2\beta}{1-k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta}.$$

Thus, if we put

$$\tan A=ik\operatorname{sn}^2\alpha,\quad \tan B=ik\operatorname{sn}^2\beta,$$

then $\tan(A-B)$ becomes $ik\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta)$.

It follows therefore that in any trigonometrical formula involving only the tangents of differences of arbitrary quantities A, B, C, &c., we may replace

$$\tan(A-B),\ \tan(A-C),\ \tan(B-C),\ \&c.$$

by

$$ik\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta),\ ik\operatorname{sn}(\alpha-\gamma)\operatorname{sn}(\alpha+\gamma),$$

$$ik\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma),\ \&c.,$$

where α, β, γ, &c. are arbitrary quantities.

Taking, for example, the formula

$$\tan(B-C)+\tan(C-A)+\tan(A-B)$$
$$=\tan(B-C)\tan(C-A)\tan(A-B)\ldots\ldots\ldots\ldots\ldots(2),$$

we deduce that

$$ik\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)+ik\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\gamma+\alpha)+ik\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta)$$
$$=-ik^3\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\gamma+\alpha)\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta)$$

viz.

$$\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)+\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\gamma+\alpha)+\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta)$$
$$+k^2\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\gamma+\alpha)\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\alpha+\beta)=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which is another form of Jacobi's equation (1).

The principle of the transformation is perhaps best seen by observing that (2) is equivalent to the algebraic identity

$$\frac{y-z}{1+yz}+\frac{z-x}{1-zx}+\frac{x-y}{1+xy}=\frac{y-z}{1+yz}\cdot\frac{z-x}{1+zx}\cdot\frac{x-y}{1+xy}.$$

Replacing x, y, z by ikx, iky, ikz, this identity becomes

$$ik\frac{y-z}{1-k^2yz}+ik\frac{z-x}{1-k^2zx}+ik\frac{x-y}{1-k^2xy}$$

$$=ik\frac{y-z}{1-k^2yz}\cdot ik\frac{z-x}{1-k^2zx}\cdot ik\frac{x-y}{1-k^2xy},$$

which, on putting

$$x=\mathrm{sn}^2\alpha,\quad y=\mathrm{sn}^2\beta,\quad z=\mathrm{sn}^2\gamma,$$

gives (3).

§ 3. The arguments may be transformed by putting

$$\beta-\gamma=y-z,$$
$$\beta+\gamma=w-x,$$
$$\gamma-\alpha=z-x,$$
$$\gamma+\alpha=w-y,$$
$$\alpha-\beta=x-y,$$
$$\alpha+\beta=w-z,$$

where x, y, z, w are arbitrary. If we take $w=x+y+z$, we reproduce the original arguments, and if we put $w=0$ we obtain the six arguments

$$y-z,\ -x,\ z-x,\ -y,\ x-y,\ -z.$$

In any formula therefore, involving the tangents of the differences of the three quantities A, B, C, we may replace

$$\tan(B-C),\ \tan(C-A),\ \tan(A-B),$$

by

$$-ik\,\mathrm{sn}\,x\,\mathrm{sn}(y-z),\ -ik\,\mathrm{sn}\,y\,\mathrm{sn}(z-x),\ -ik\,\mathrm{sn}\,z\,\mathrm{sn}(x-y),$$

where x, y, z are arbitrary; and since the signs of the arguments of the tangents may be changed throughout at pleasure, we see that there is no occasion to retain the negative signs prefixed to the expressions last written.

Thus from (2) we deduce that

$$\operatorname{sn}x \operatorname{sn}(y-z) + \operatorname{sn}y \operatorname{sn}(z-x) + \operatorname{sn}z \operatorname{sn}(x-y)$$
$$+ k^2 \operatorname{sn}x \operatorname{sn}y \operatorname{sn}z \operatorname{sn}(y-z) \operatorname{sn}(z-x) \operatorname{sn}(x-y) = 0,$$

which is another form of (1) and (3).

§ 4. Writing A, B, C in place of x, y, z, it has been shown in the last section that if p, q, r are any three quantities whose sum is zero, then in any formula involving $\tan p$, $\tan q$, $\tan r$, we may replace these tangents by

$$ik \operatorname{sn}A \operatorname{sn}(B-C),\ ik \operatorname{sn}B \operatorname{sn}(C-A),\ ik \operatorname{sn}C \operatorname{sn}(A-B),$$

where A, B, C are unrestricted.

Since A, B, C are arbitrary we may substitute

$$K+A, \ldots \text{ or } iK'+A \ldots \text{ or } K+iK'+A, \ldots \text{ for } A, B, C,$$

and it therefore follows that we may replace $\tan p$ not only by $ik \operatorname{sn}A \operatorname{sn}(B-C)$, but also by

$$i\frac{\operatorname{sn}(B-C)}{\operatorname{sn}A},$$

$$\text{or } ik\frac{\operatorname{cn}A \operatorname{sn}(B-C)}{\operatorname{dn}A},$$

$$\text{or } i\frac{\operatorname{dn}A \operatorname{sn}(B-C)}{\operatorname{cn}A},$$

if $\tan q$ and $\tan r$ be also replaced by the corresponding expressions.

The modulus k is also arbitrary, and by transforming it into k', $\frac{1}{k'}$, &c., we see that we may replace $\tan p$ by any one of the twelve quantities

$$ik \operatorname{sn}A \operatorname{sn}(B-C),\quad ik'\frac{\operatorname{sn}A \operatorname{sn}(B-C)}{\operatorname{cn}A \operatorname{cn}(B-C)},\quad kk'\frac{\operatorname{sn}A \operatorname{sn}(B-C)}{\operatorname{dn}A \operatorname{dn}(B-C)},$$

$$i\frac{\operatorname{sn}(B-C)}{\operatorname{sn}A},\quad i\frac{\operatorname{cn}A \operatorname{sn}(B-C)}{\operatorname{sn}A \operatorname{cn}(B-C)},\quad i\frac{\operatorname{dn}A \operatorname{sn}(B-C)}{\operatorname{sn}A \operatorname{dn}(B-C)},$$

$$ik\frac{\operatorname{cn}A \operatorname{sn}(B-C)}{\operatorname{dn}A},\quad \frac{\operatorname{dn}A \operatorname{sn}(B-C)}{\operatorname{cn}(B-C)},\quad k\frac{\operatorname{cn}A \operatorname{sn}(B-C)}{\operatorname{dn}(B-C)},$$

$$i\frac{\operatorname{dn}A \operatorname{sn}(B-C)}{\operatorname{cn}A},\quad k'\frac{\operatorname{sn}(B-C)}{\operatorname{dn}A \operatorname{cn}(B-C)},\quad ik'\frac{\operatorname{sn}(B-C)}{\operatorname{cn}A \operatorname{dn}(B-C)},$$

if $\tan q$ and $\tan r$ be also replaced by the corresponding expressions involving B and $C-A$, and C and $A-B$.

Using the functional signs sc, cd, ns, &c., where $\operatorname{sc} u$, $\operatorname{cd} u$, $\operatorname{ns} u$, &c. denote $\frac{\operatorname{sn} u}{\operatorname{cn} u}$, $\frac{\operatorname{cn} u}{\operatorname{dn} u}$, $\frac{1}{\operatorname{sn} u}$, &c., these twelve expressions may be written

$ik \operatorname{sn} A \operatorname{sn}(B-C)$, $ik' \operatorname{sc} A \operatorname{sc}(B-C)$, $kk' \operatorname{sd} A \operatorname{sd}(B-C)$,

$i \operatorname{ns} A \operatorname{sn}(B-C)$, $i \operatorname{cs} A \operatorname{sc}(B-C)$, $i \operatorname{ds} A \operatorname{sd}(B-C)$,

$ik \operatorname{cd} A \operatorname{sn}(B-C)$, $\operatorname{dn} A \operatorname{sc}(B-C)$, $k \operatorname{cn} A \operatorname{sd}(B-C)$,

$i \operatorname{dc} A \operatorname{sn}(B-C)$, $k' \operatorname{nd} A \operatorname{sc}(B-C)$, $ik' \operatorname{nc} A \operatorname{sd}(B-C)$.

§ 5. Taking, as an example, the trigonometrical formula (2), viz.

$$\tan p + \tan q + \tan r = \tan p \tan q \tan r \quad \ldots\ldots(2),$$

we deduce from it the twelve formulæ

$$\Sigma \operatorname{sn} A \operatorname{sn}(B-C) + k^2 \Pi \operatorname{sn} A \operatorname{sn}(B-C) = 0,$$
$$\Sigma \operatorname{ns} A \operatorname{sn}(B-C) + \Pi \operatorname{ns} A \operatorname{sn}(B-C) = 0,$$
$$\Sigma \operatorname{cd} A \operatorname{sn}(B-C) + k^2 \Pi \operatorname{cd} A \operatorname{sn}(B-C) = 0,$$
$$\Sigma \operatorname{dc} A \operatorname{sn}(B-C) + \Pi \operatorname{dc} A \operatorname{sn}(B-C) = 0,$$

$$\Sigma \operatorname{sc} A \operatorname{sc}(B-C) + k'^2 \Pi \operatorname{sc} A \operatorname{sc}(B-C) = 0,$$
$$\Sigma \operatorname{cs} A \operatorname{sc}(B-C) + \Pi \operatorname{cs} A \operatorname{sc}(B-C) = 0,$$
$$\Sigma \operatorname{dn} A \operatorname{sc}(B-C) - \Pi \operatorname{dn} A \operatorname{sc}(B-C) = 0,$$
$$\Sigma \operatorname{nd} A \operatorname{sc}(B-C) - k'^2 \Pi \operatorname{nd} A \operatorname{sc}(B-C) = 0,$$

$$\Sigma \operatorname{sd} A \operatorname{sd}(B-C) - k^2 k'^2 \Pi \operatorname{sd} A \operatorname{sd}(B-C) = 0,$$
$$\Sigma \operatorname{ds} A \operatorname{sd}(B-C) + \Pi \operatorname{ds} A \operatorname{sd}(B-C) = 0,$$
$$\Sigma \operatorname{cn} A \operatorname{sd}(B-C) - k^2 \Pi \operatorname{cn} A \operatorname{sd}(B-C) = 0,$$
$$\Sigma \operatorname{nc} A \operatorname{sd}(B-C) + k'^2 \Pi \operatorname{nc} A \operatorname{sd}(B-C) = 0;$$

where Σ denotes the sum of the three expressions, and Π their product, *e.g.* the last equation written at length is

$$\operatorname{nc} A \operatorname{sd}(B-C) + \operatorname{nc} B \operatorname{sd}(C-A) + \operatorname{nc} C \operatorname{sd}(A-B)$$
$$+ k'^2 \operatorname{nc} A \operatorname{nc} B \operatorname{nc} C \operatorname{sd}(B-C) \operatorname{sd}(C-A) \operatorname{sd}(A-B) = 0.$$

§ 6. If s, c, d denote sn, cn, dn, and if we use the suffixes 1, 2... 6 to refer respectively to the six arguments

$$A,\ B-C,\ B,\ C-A,\ C,\ A-B,$$

these twelve formulæ may be written

$$\begin{aligned}
&s_1s_2 \quad + s_3s_4 \quad + s_5s_6 \quad + k^2s_1s_2s_3s_4s_5s_6 = 0,\\
&s_2s_3s_5 \quad + s_4s_5s_1 \quad + s_6s_1s_3 \quad + s_2s_4s_6 \quad = 0,\\
&s_2c_1d_3d_5 + s_4c_3d_5d_1 + s_6c_5d_1d_3 + k^2s_2s_4s_6c_1c_3c_5 = 0,\\
&s_2d_1c_3c_5 + s_4d_3c_5c_1 + s_6d_5c_1c_3 + s_2s_4s_6d_1d_3d_5 = 0,
\end{aligned}$$

$$\begin{aligned}
&s_1s_2c_3c_4c_5c_6 + s_3s_4c_5c_6c_1c_2 + s_5s_6c_1c_2c_3c_4 + k'^2s_1s_2s_3s_4s_5s_6 = 0,\\
&s_2s_3s_5c_1c_4c_6 + s_4s_5s_1c_3c_6c_2 + s_6s_1s_3c_5c_2c_4 + \quad s_2s_4s_6c_1c_3c_5 = 0,\\
&s_2d_1c_4c_6 \quad + s_4d_3c_6c_2 \quad + s_6d_5c_2c_4 \quad - \quad s_2s_4s_6d_1d_3d_5 = 0,\\
&s_2d_3d_5c_4c_6 + s_4d_5d_1c_6c_2 + s_6d_1d_3c_2c_4 - k'^2s_2s_4s_6 \quad = 0,
\end{aligned}$$

$$\begin{aligned}
&s_1s_2d_3d_4d_5d_6 + s_3s_4d_5d_6d_1d_2 + s_5s_6d_1d_2d_3d_4 - k^2k'^2s_1s_2s_3s_4s_5s_6 = 0,\\
&s_2s_3s_5d_1d_4d_6 + s_4s_5s_1d_3d_6d_2 + s_6s_1s_3d_5d_2d_4 + \quad s_2s_4s_6d_1d_3d_5 = 0,\\
&s_2c_1d_4d_6 \quad + s_4c_3d_6d_2 \quad + s_6c_5d_2d_4 \quad - k^2 \quad s_2s_4s_6c_1c_3c_5 = 0,\\
&s_2c_3c_5d_4d_6 + s_4c_5c_1d_6d_2 + s_6c_1c_3d_2d_4 + k'^2 \quad s_2s_4s_6 = 0.
\end{aligned}$$

§ 7. If instead of (2) we start with the formula

$$\tan 2p + \tan 2q + \tan 2r = \tan 2p \tan 2q \tan 2r,$$

that is

$$\frac{\tan p}{1-\tan^2 p} + \frac{\tan q}{1-\tan^2 q} + \frac{\tan r}{1-\tan^2 r} = 4\,\frac{\tan p \tan q \tan r}{(1-\tan^2 p)(1-\tan^2 q)(1-\tan^2 r)},$$

we deduce from it that

$$\sum \frac{\operatorname{sn} A \operatorname{sn}(B-C)}{1+k^2\operatorname{sn}^2 A \operatorname{sn}^2(B-C)} + 4k^2 \prod \frac{\operatorname{sn} A \operatorname{sn}(B-C)}{1+k^2\operatorname{sn}^2 A \operatorname{sin}^2(B-C)} = 0,$$

$$\sum \frac{\operatorname{sn} A \operatorname{sn}(B-C)}{\operatorname{sn}^2 A + \operatorname{sn}^2(B-C)} + 4 \prod \frac{\operatorname{sn} A \operatorname{sn}(B-C)}{\operatorname{sin}^2 A + \operatorname{sn}^2(B-C)} = 0,$$

&c. &c.

§ 8. It follows from § 3 that instead of the arguments A, $B-C$, B, ... we may use either of the sets of arguments placed opposite to them in the following scheme:

$$\begin{array}{lll} A\ , & \beta-\gamma, & y-z, \\ B-C, & \beta+\gamma, & w-x, \\ B\ , & \gamma-\alpha, & z-x, \\ C-A, & \gamma+\alpha, & w-y, \\ C\ , & \alpha-\beta, & x-y, \\ A-B, & \alpha+\beta, & w-z. \end{array}$$

Thus, for example, the last equation in the preceding section may be written

$$\sum \frac{\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)}{\operatorname{sn}^2(\beta-\gamma)+\operatorname{sn}^2(\beta+\gamma)} + 4\prod \frac{\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)}{\operatorname{sn}^2(\beta-\gamma)+\operatorname{sn}^2(\beta+\gamma)} = 0,$$

whence

$$\sum \frac{\operatorname{sn}^2\beta\operatorname{cn}^2\gamma\operatorname{dn}^2\gamma - \operatorname{sn}^2\gamma\operatorname{cn}^2\beta\operatorname{dn}^2\beta}{\operatorname{sn}^2\beta\operatorname{cn}^2\gamma\operatorname{dn}^2\gamma + \operatorname{sn}^2\gamma\operatorname{cn}^2\beta\operatorname{dn}^2\beta} + \prod \frac{\operatorname{sn}^2\beta\operatorname{cn}^2\gamma\operatorname{dn}^2\gamma - \operatorname{sn}^2\gamma\operatorname{cn}^2\beta\operatorname{dn}^2\beta}{\operatorname{sn}^2\beta\operatorname{cn}^2\gamma\operatorname{dn}^2\gamma + \operatorname{sn}^2\gamma\operatorname{cn}^2\beta\operatorname{dn}^2\beta} = 0.$$

§ 9. If we put

$$\tan p = ik\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma),$$

we have $$\sec^2 p = 1 - k^2\operatorname{sn}^2(\beta-\gamma)\operatorname{sn}^2(\beta+\gamma),$$

and therefore $$\cos p = \frac{1}{\sqrt{\{1-k^2\operatorname{sn}^2(\beta-\gamma)\operatorname{sn}^2(\beta+\gamma)\}}},$$

and $$\sin p = \frac{ik\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\beta+\gamma)}{\sqrt{\{1-k^2\operatorname{sn}^2(\beta-\gamma)\operatorname{sn}^2(\beta+\gamma)\}}}.$$

Now if A', B', C' are the angles of a triangle, the three quantities A', B', $C'-180°$ are such that their sum is zero, and we may therefore put

$$A' = p, \quad B' = q, \quad C' - 180° = r,$$

so that $$\sin r = -\sin C', \quad \cos r = -\cos C'.$$

In any trigonometrical formula therefore involving functions of A', B', C', the angles of a triangle, we may put

$$\sin A' = \frac{ik\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma)}{\sqrt{\{1-k^2\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\beta+\gamma)\}}} = \frac{ik\,(\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)}{\sqrt{\{(1-k^2\mathrm{sn}^4\beta)(1-k^2\mathrm{sn}^4\gamma)\}}},$$

$$\cos A' = \frac{1}{\sqrt{\{1-k^2\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\beta+\gamma)\}}} = \frac{1-k^2\,\mathrm{sn}^2\beta\,\mathrm{sn}^2\gamma}{\sqrt{\{(1-k^2\mathrm{sn}^4\beta)(1-k^2\mathrm{sn}^4\gamma)\}}};$$

making similar substitutions for $\sin B'$, $\cos B'$, $\sin C'$, $\cos C'$ but prefixing the negative sign in the case of the two latter. We thus have

$$\sin 2A' = \frac{2ik\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma)}{1-k^2\,\mathrm{sn}^2\,(\beta-\gamma)\,\mathrm{sn}^2\,(\beta+\gamma)},$$

$$\cos 2A' = \frac{1+k^2\,\mathrm{sn}^2\,(\beta-\gamma)\,\mathrm{sn}^2\,(\beta+\gamma)}{1-k^2\,\mathrm{sn}^2\,(\beta-\gamma)\,\mathrm{sn}^2\,(\beta+\gamma)},$$

and it may be observed that

$$\cos A' + i\sin A' = \frac{1-k\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma)}{\sqrt{\{1-k^2\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma)\}}},$$

whence $$e^{iA'} = \sqrt{\left\{\frac{1-k\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma)}{1+k\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma)}\right\}},$$

and therefore

$$A' = \frac{1}{2i}\log\left\{\frac{(1-k\,\mathrm{sn}^2\beta)\,(1+k\,\mathrm{sn}^2\gamma)}{(1+k\,\mathrm{sn}^2\beta)\,(1-k\,\mathrm{sn}^2\gamma)}\right\}.$$

For example, from the formula

$$\sin^2 2A' + \sin^2 2B' + \sin^2 2C' + 2\cos 2A'\cos 2B'\cos 2C' = 2,$$

we deduce that

$$1+2k^2\sum\frac{\mathrm{sn}^2A\,\mathrm{sn}^2(B-C)}{\{1-k^2\,\mathrm{sn}^2A\,\mathrm{sn}^2(B-C)\}^2} = \prod\frac{1+k^2\mathrm{sn}^2A\,\mathrm{sn}^2(B-C)}{1-k^2\mathrm{sn}^2A\,\mathrm{sn}^2(B-C)}.$$

Algebraical formulæ, § 10.

§ 10. It may be remarked that since we may substitute for $\tan p$ either

$$ik\,\mathrm{sn}\,(\beta-\gamma)\,\mathrm{sn}\,(\beta+\gamma),$$

or $$ik'\,\mathrm{sc}\,(\beta-\gamma)\,\mathrm{sc}\,(\beta+\gamma),$$

or $$kk'\,\mathrm{sd}\,(\beta-\gamma)\,\mathrm{sd}\,(\beta+\gamma),$$

it follows that in any formula involving $\tan p$, $\tan q$, $\tan r$ we may substitute for $\tan p$ not only

$$ik\,\frac{x-y}{1-k^2xy},$$

but also

$$ik'\,\frac{x-y}{1-x-y+k^2xy},$$

or

$$kk'\,\frac{x-y}{1-k^2x-k^2y-k^2xy},$$

making similar substitutions for $\tan q$ and $\tan r$.

If we put $k=-1$, so that $k'=\sqrt{2}$, the two last-written expressions become

$$\frac{\sqrt{2}.(x-y)}{1-x-y-xy} \quad \text{and} \quad \frac{\sqrt{2}.(x-y)}{1+x+y-xy}.$$

Thus from (2) we deduce not only that

$$\sum\frac{y-z}{1+yz}=\prod\frac{y-z}{1+yz},$$

but also that

$$\sum\frac{y-z}{1-y-z+k^2yz}+k'^2\prod\frac{y-z}{1-y-z+k^2yz}=0,$$

$$\sum\frac{y-z}{1-k^2y-k^2z+k^2yz}-k^2k'^2\prod\frac{y-z}{1-k^2y-k^2z+k^2yz}=0,$$

which may be verified without difficulty.

Putting $k=-1$, these identities may be written

$$\Sigma(y-z)(1-z-x-zx)(1-x-y-xy)$$
$$+2(y-z)(z-x)(x-y)=0,$$

$$\Sigma(y-z)(1+z+x-zx)(1+x+y-xy)$$
$$+2(y-z)(z-x)(x-y)=0.$$

Formulæ involving sn*'s only. Derivation of elliptic function formulæ from one another*, §§11–16.

§11. If $\operatorname{sn} u$, when the modulus is put equal to unity, be denoted by $\operatorname{sn}_1 u$, we have

$$\operatorname{sn}_1(u-v)=\frac{\operatorname{sn}_1 u-\operatorname{sn}_1 v}{1-\operatorname{sn}_1 u\operatorname{sn}_1 v},$$

and also, the modulus being k,

$$k\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\alpha+\beta)=\frac{k\,\mathrm{sn}^2\alpha-k\,\mathrm{sn}^2\beta}{1-k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta}.$$

We therefore have the theorem that in a formula involving only the sn's of differences of quantities, if we first put $k=1$, we may replace $\mathrm{sn}(u-v)$ by $k\,\mathrm{sn}(u-v)\,\mathrm{sn}(u+v)$, and so on throughout for each sn of a difference, thus obtaining a new formula which will be true for modulus k.

Thus, taking Jacobi's formula (1), viz.

$$\mathrm{sn}(w-x)\,\mathrm{sn}(y-z)+\mathrm{sn}(w-y)\,\mathrm{sn}(z-x)+\mathrm{sn}(w-z)\,\mathrm{sn}(x-y)$$
$$+k^2\,\mathrm{sn}(w-x)\,\mathrm{sn}(w-y)\,\mathrm{sn}(w-z)\,\mathrm{sn}(y-z)\,\mathrm{sn}(z-x)\,\mathrm{sn}(x-y)=0,$$

we put $k=1$, giving

$$\mathrm{sn}_1(w-x)\,\mathrm{sn}_1(y-z)+\mathrm{sn}_1(w-y)\,\mathrm{sn}_1(z-x)+\mathrm{sn}_1(w-z)\,\mathrm{sn}_1(x-y)$$
$$+\mathrm{sn}_1(w-x)\,\mathrm{sn}_1(w-y)\,\mathrm{sn}_1(w-z)\,\mathrm{sn}_1(y-z)\,\mathrm{sn}_1(z-x)\,\mathrm{sn}_1(x-y)=0,$$

whence, replacing $\mathrm{sn}_1(w-x)$ by $k\,\mathrm{sn}(w-x)\,\mathrm{sn}(w+x)$, $\mathrm{sn}_1(y-z)$ by $k\,\mathrm{sn}(y-z)\,\mathrm{sn}(y+z)$, &c., we find

$$\begin{aligned}
&\mathrm{sn}(w-x)\,\mathrm{sn}(w+x)\,\mathrm{sn}(y-z)\,\mathrm{sn}(y+z)\\
+\;&\mathrm{sn}(w-y)\,\mathrm{sn}(w+y)\,\mathrm{sn}(z-x)\,\mathrm{sn}(z+x)\\
+\;&\mathrm{sn}(w-z)\,\mathrm{sn}(w+z)\,\mathrm{sn}(x-y)\,\mathrm{sn}(x+y)\\
+\;&k^4\,\mathrm{sn}(w-x)\,\mathrm{sn}(w+x)\,\mathrm{sn}(w-y)\,\mathrm{sn}(w+y)\,\mathrm{sn}(w-z)\,\mathrm{sn}(w+z)\\
&\times\mathrm{sn}(y-z)\,\mathrm{sn}(y+z)\,\mathrm{sn}(z-x)\,\mathrm{sn}(z+x)\,\mathrm{sn}(x-y)\,\mathrm{sn}(x+y)=0.
\end{aligned}$$

§ 12. In place of the six arguments

$$A,\ B-C,\ B,\ C-A,\ C,\ A-B,$$

we may, as remarked in § 8, write

$$y-z,\ w-x,\ z-x,\ w-y,\ x-y,\ w-z;$$

and it is to be observed that we may replace the twelve arguments $y-z$, $y+z$, &c., by their equivalents in the following system of equations,

$$\begin{aligned}
y-z&=\beta-\gamma, & y+z&=\delta-\alpha,\\
z-x&=\gamma-\alpha, & z+x&=\delta-\beta,\\
x-y&=\alpha-\beta, & x+y&=\delta-\gamma,\\
w-x&=\beta+\gamma, & w+x&=\delta+\alpha,\\
w-y&=\gamma+\alpha, & w+y&=\delta+\beta,\\
w-z&=\alpha+\beta, & w+z&=\delta+\beta,
\end{aligned}$$

In any formula therefore involving the three arguments $y-z$, $z-x$, $x-y$, or the six arguments $y-z$, $z-x$, $x-y$, $w-x$, $w-y$, $w-z$, we may replace

$\mathrm{sn}_1(y-z)$ by $k\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\delta-\alpha)$ or $-k\,\mathrm{sn}\,\alpha\,\mathrm{sn}(\beta-\gamma)$,

$\mathrm{sn}_1(z-x)$ „ $k\,\mathrm{sn}(\gamma-\alpha)\,\mathrm{sn}(\delta-\beta)$ „ $-k\,\mathrm{sn}\,\beta\,\mathrm{sn}(\gamma-\alpha)$,

$\mathrm{sn}_1(x-y)$ „ $k\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\delta-\gamma)$ „ $-k\,\mathrm{sn}\,\gamma\,\mathrm{sn}(\alpha-\beta)$,

$\mathrm{sn}_1(w-x)$ „ $k\,\mathrm{sn}(\beta+\gamma)\,\mathrm{sn}(\delta+\alpha)$ „ $k\,\mathrm{sn}\,\alpha\,\mathrm{sn}(\beta+\gamma)$,

$\mathrm{sn}_1(w-y)$ „ $k\,\mathrm{sn}(\gamma+\alpha)\,\mathrm{sn}(\delta+\beta)$ „ $k\,\mathrm{sn}\,\beta\,\mathrm{sn}(\gamma+\alpha)$,

$\mathrm{sn}_1(w-z)$ „ $k\,\mathrm{sn}(\alpha+\beta)\,\mathrm{sn}(\delta+\gamma)$ „ $k\,\mathrm{sn}\,\gamma\,\mathrm{sn}(\alpha+\beta)$.

the second set of values being deducible from the first by putting $\delta=0$.

It follows also from these equations that we may replace

$\mathrm{sn}_1 A\,\mathrm{sn}_1(B-C)$ by $k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\beta+\gamma)$,

$\mathrm{sn}_1 B\,\mathrm{sn}_1(C-A)$ „ $k^2\,\mathrm{sn}^2\beta\,\mathrm{sn}(\gamma-\alpha)\,\mathrm{sn}(\gamma+\alpha)$,

$\mathrm{sn}_1 C\,\mathrm{sn}_1(A-B)$ „ $k^2\,\mathrm{sn}^2\gamma\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\alpha+\beta)$.

§ 13. The second of the twelve formulæ in § 5, viz.

$$\Sigma\,\mathrm{ns}\,A\,\mathrm{sn}(B-C)+\Pi\,\mathrm{ns}\,A\,\mathrm{sn}(B-C)=0,$$

may be written

$$\mathrm{sn}(w-y)\,\mathrm{sn}(w-z)\,\mathrm{sn}(y-z)+\mathrm{sn}(w-z)\,\mathrm{sn}(w-x)\,\mathrm{sn}(z-x)$$
$$+\mathrm{sn}(w-x)\,\mathrm{sn}(w-y)\,\mathrm{sn}(x-y)+\mathrm{sn}(y-z)\,\mathrm{sn}(z-x)\,\mathrm{sn}(x-y)=0;$$

whence, by the last section, we find

$$\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma+\alpha)\,\mathrm{sn}(\alpha+\beta)\,\mathrm{sn}(\delta-\alpha)\,\mathrm{sn}(\delta+\beta)\,\mathrm{sn}(\delta+\gamma)$$
$$+\mathrm{sn}(\beta+\gamma)\,\mathrm{sn}(\gamma-\alpha)\,\mathrm{sn}(\alpha+\beta)\,\mathrm{sn}(\delta+\alpha)\,\mathrm{sn}(\delta-\beta)\,\mathrm{sn}(\delta+\gamma)$$
$$+\mathrm{sn}(\beta+\gamma)\,\mathrm{sn}(\gamma+\alpha)\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\delta+\alpha)\,\mathrm{sn}(\delta+\beta)\,\mathrm{sn}(\delta-\gamma)$$
$$+\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\delta-\alpha)\,\mathrm{sn}(\delta-\beta)\,\mathrm{sn}(\delta-\gamma)=0.$$

If we put $w=0$ in the formula proved in § 11 and replace x, y, z by α, β, γ we obtain the equation

$$\mathrm{sn}^2\alpha\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\beta+\gamma)$$
$$+\mathrm{sn}^2\beta\,\mathrm{sn}(\gamma-\alpha)\,\mathrm{sn}(\gamma+\alpha)$$
$$+\mathrm{sn}^2\gamma\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\alpha+\beta)$$
$$+k^4\mathrm{sn}^2\alpha\mathrm{sn}^2\beta\mathrm{sn}^2\gamma\,\mathrm{sn}(\beta-\gamma)\mathrm{sn}(\beta+\gamma)\mathrm{sn}(\gamma-\alpha)\mathrm{sn}(\gamma+\alpha)\mathrm{sn}(\alpha-\beta)\mathrm{sn}(\alpha+\beta)$$
$$=0,$$

which is also derivable at once from the first of the group of twelve in § 5 by the substitutions given at the end of the last section.

§ 14. Several formulæ involving sn's to which the transformations of § 12 may be applied are given in the *Messenger of Mathematics*, vol. x., pp. 92–95. It is there shown, for example, that if $s_1, s_2, \ldots s_6$ denote the sn's of the six arguments A, $B-C$, B, $C-A$, C, $A-B$, then

$$s_1s_4s_6(s_2^2+s_3^2+s_5^2)+s_3s_6s_2(s_4^2+s_5^2+s_1^2)$$
$$+s_5s_2s_4(s_6^2+s_1^2+s_3^2)+s_1s_3s_5(s_2^2+s_4^2+s_6^2)=0,$$

and
$$s_2s_3s_5(s_2^2+s_3^2+s_5^2)+s_4s_5s_1(s_4^2+s_5^2+s_1^2)$$
$$+s_6s_1s_3(s_6^2+s_1^2+s_3^2)+s_2s_4s_6(s_2^2+s_4^2+s_6^2)$$
$$+(s_1s_2+s_3s_4+s_5s_6)(s_1s_4s_6+s_3s_6s_2+s_5s_2s_4+s_1s_3s_5)=0.$$

We see therefore that these formulæ are still true if $s_1, s_2, \ldots s_6$ denote the expressions given by the equations

$$s_1=\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\delta-\alpha),\quad s_2=\operatorname{sn}(\beta+\gamma)\operatorname{sn}(\delta+\alpha),$$
$$s_3=\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\delta-\beta),\quad s_4=\operatorname{sn}(\gamma+\alpha)\operatorname{sn}(\delta+\beta),$$
$$s_5=\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\delta-\gamma),\quad s_6=\operatorname{sn}(\alpha+\beta)\operatorname{sn}(\delta+\gamma).$$

§ 15. Formulæ involving sn's have been alone considered in the four preceding sections; but it is scarcely necessary to remark that corresponding formulæ involving sc's and sd's may be derived from them by changing the modulus from k to k' and $\frac{ik}{k'}$, since as the result of these changes we may replace $\operatorname{sn}u$ by $i\operatorname{sc}u$ and $k'\operatorname{sd}u$ respectively. Thus, for example, we have

$$\operatorname{sc}(\beta-\gamma)\operatorname{sc}(\gamma+\alpha)\operatorname{sc}(\alpha+\beta)\operatorname{sc}(\delta-\alpha)\operatorname{sc}(\delta+\beta)\operatorname{sc}(\delta+\gamma)$$
$$+\operatorname{sc}(\beta+\gamma)\operatorname{sc}(\gamma-\alpha)\operatorname{sc}(\alpha+\beta)\operatorname{sc}(\delta+\alpha)\operatorname{sc}(\delta-\beta)\operatorname{sc}(\delta+\gamma)$$
$$+\operatorname{sc}(\beta+\gamma)\operatorname{sc}(\gamma+\alpha)\operatorname{sc}(\alpha-\beta)\operatorname{sc}(\delta+\alpha)\operatorname{sc}(\delta+\beta)\operatorname{sc}(\delta-\gamma)$$
$$+\operatorname{sc}(\beta-\gamma)\operatorname{sc}(\gamma-\alpha)\operatorname{sc}(\alpha-\beta)\operatorname{sc}(\gamma-\alpha)\operatorname{sc}(\delta-\beta)\operatorname{sc}(\delta-\gamma)=0,$$

and a similar equation involving sd's.

§ 16. It may be noticed that the formula

$$\mathrm{sn}_1(u-v) = \frac{\mathrm{sn}_1 u - \mathrm{sn}_1 v}{1 - \mathrm{sn}_1 u \,\mathrm{sn}_1 v},$$

is in fact equivalent to

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

for $\mathrm{sn}_1 u = -i \tan iu$; so that we may pass from a formula involving sn's to one involving tangents by putting $k=1$, multiplying the arguments throughout by i, and replacing $\mathrm{sn}\, iu$, &c., by $i \tan u$, &c.

Thus from Jacobi's formula (1) we deduce that

$$\tan(w-x)\tan(y-z) + \tan(w-y)\tan(z-x) + \tan(w-z)\tan(x-y)$$
$$+ \tan(w-x)\tan(w-y)\tan(w-z)\tan(y-z)\tan(z-x)\tan(x-y) = 0,$$

which by § 2 gives the formulæ involving sn's obtained in § 11.

It may be remarked that the formula

$$\mathrm{sn}_1(y-z) + \mathrm{sn}_1(z-x) + \mathrm{sn}_1(x-y) + \mathrm{sn}_1(y-z)\,\mathrm{sn}_1(z-x)\,\mathrm{sn}_1(x-y) = 0$$

is the particular case, corresponding to $k=1$, of the formula

$$E(u) + E(v) - E(u+v) = k^2 \,\mathrm{sn}\, u \,\mathrm{sn}\, v \,\mathrm{sn}\,(u+v),$$

for, when $k=1$, we have $E(u) = \mathrm{sn}_1 u$.

Forms of the twelve arguments, § 17.

§ 17. Jacobi, in the case of six arguments, put

$$w-x=a, \quad x-y=u, \quad y-z=b,$$

giving

$$\begin{aligned} y-z &= b, & w-x &= a, \\ z-x &= -(u+b), & w-y &= u+a, \\ x-y &= u, & w-z &= u+a+b; \end{aligned}$$

if we have, in addition, the six arguments in which the terms of the binomials are connected by the positive sign, we may put $w+z=v$, whence

$$\begin{aligned} y+z &= -(u-v+a), & w+x &= u+v+b, \\ z+x &= v-a, & w+y &= v+b, \\ x+y &= v-a+b, & w+z &= v. \end{aligned}$$

Another system of arguments which is worth notice may be obtained by putting $y-z$, $y+z$, $w-x$, $w+x$ equal to a, b, c, $-d$ respectively.

For convenience of reference the different systems of arguments are collected together in the following list:

$$\begin{aligned}
y-z=\beta-\gamma&= b &&= a,\\
z-x=\gamma-\alpha&=-(u+b) &&= a',\\
x-y=\alpha-\beta&= u &&=-a'',\\
\\
y+z=\delta-\alpha&=-(u-v+a) &&= b,\\
z+x=\delta-\beta&= v-a &&=-b',\\
x+y=\delta-\gamma&= v-a+b &&= b'',\\
\\
w-x=\beta+\gamma&= a &&= c,\\
w-y=\gamma+\alpha&= u+a &&=-c',\\
w-z=\alpha+\beta&= u+a+b &&= c'',\\
\\
w+x=\delta+\alpha&= u+v+b &&=-d,\\
w+y=\delta+\beta&= v+b &&= d',\\
w+z=\delta+\gamma&= v &&=-d'',
\end{aligned}$$

where in the system of arguments in the last column a, b, c, d are arbitrary, and

$$\begin{aligned}
a'&=\tfrac{1}{2}(-a+b+c+d), & a''&=\tfrac{1}{2}(a+b+c+d),\\
b'&=\tfrac{1}{2}(\ a-b+c+d), & b''&=\tfrac{1}{2}(a+b-c-d),\\
c'&=\tfrac{1}{2}(\ a+b-c+d), & c''&=\tfrac{1}{2}(a-b+c-d),\\
d'&=\tfrac{1}{2}(\ a+b+c-d), & d''&=\tfrac{1}{2}(a-b-c+d).
\end{aligned}$$

Trigonometrical formulæ, § 18.

§ 18. If we take an elliptic function formula involving sn's only, such as (1), we may derive trigonometrical formulæ from it in two ways, viz. (i) by putting $k=0$, we have

$$\sin(w-x)\sin(y-z)+\sin(w-y)\sin(z-x)+\sin(w-z)\sin(x-y)=0\ldots(5),$$

and, by putting $k = 1$,

$$\mathrm{sn}_1(w-x)\,\mathrm{sn}_1(y-z) + \mathrm{sn}_1(w-y)\,\mathrm{sn}_1(z-x) + \mathrm{sn}_1(w-z)\,\mathrm{sn}_1(x-y)$$
$$+ \mathrm{sn}_1(w-x)\,\mathrm{sn}_1(w-y)\,\mathrm{sn}_1(w-z)\,\mathrm{sn}_1(y-z)\,\mathrm{sn}_1(z-x)\,\mathrm{sn}_1(x-y)$$
$$= 0,$$

whence, by § 16,

$$\tan(w-x)\tan(y-z) + \tan(w-y)\tan(z-x) + \tan(w-z)\tan(x-y)$$
$$+ \tan(w-x)\tan(w-y)\tan(w-z)\tan(y-z)\tan(z-x)\tan(x-y)$$
$$= 0.$$

Now, by § 11, we deduce from (1) that

$$\Sigma\,\mathrm{sn}(w-x)\,\mathrm{sn}(w+x)\,\mathrm{sn}(y-z)\,\mathrm{sn}(y+z)$$
$$+ k^4\,\Pi\,\mathrm{sn}(w-x)\,\mathrm{sn}(w+x)\,\mathrm{sn}(y-z)\,\mathrm{sn}(y+z) = 0,$$

and therefore, putting $k = 0$,

$$\Sigma \sin(w-x)\sin(w+x)\sin(y-z)\sin(y+z) = 0,$$

and, putting $k = 1$,

$$\Sigma \tan(w-x)\tan(w+x)\tan(y-z)\tan(y+z)$$
$$+ \Pi \tan(w-x)\tan(w+x)\tan(y-z)\tan(y+z) = 0.$$

The former of these last two equations may however be deduced directly from the trigonometrical formula (5), viz.

$$\Sigma \sin(w-x)\sin(y-z) = 0,$$

for, expanding the sines and considering only the terms of the lowest order, we have

$$\Sigma\,(w-x)(y-z) = 0,$$

whence, writing x^2, y^2, z^2, w^2 in place of x, y, z, w,

$$\Sigma\,(w^2-x^2)(y^2-z^2) = 0,$$

and therefore

$$\Sigma \sin(w-x)\sin(w+x)\sin(y-z)\sin(y+z) = 0,$$

since $\sin(u+v)\sin(u-v) = \sin^2 u - \sin^2 v$ which, being the difference of two squares, is of the form $u^2 - v^2$.

In general whenever we have a formula the terms of which consist of products of sines of differences,* viz.

$$\Sigma . \Pi \sin (\alpha - \beta) = 0,$$

the number of sines forming the product being the same in each term, we deduce that

$$\Sigma . \Pi (\alpha - \beta) = 0;$$

and therefore $\Sigma . \Pi (\alpha^2 - \beta^2) = 0,$

whence $\Sigma . \Pi \sin (\alpha - \beta) \sin (\alpha + \beta) = 0;$

but this method of deriving one sine-formula from another can be applied only when the number of sines multiplied together is the same in each term of the formula.

In the case of elliptic function formulæ this condition is not necessary; we put $k = 1$, and obtain a formula involving $\mathrm{sn}_1 (\alpha - \beta)$, &c.; we then replace $\mathrm{sn}_1 (\alpha - \beta)$, &c. by

$$k \,\mathrm{sn}\, (\alpha - \beta) \,\mathrm{sn}\, (\alpha - \beta), \text{ &c.},$$

and we deduce a trigonometrical formula involving sines by putting $k = 0$ and a trigonometrical formula involving tangents by putting $k = 1$ and replacing $\mathrm{sn}_1 (i\alpha - i\beta)$ by $i \tan (\alpha - \beta)$.

NOTE ON THE STANDARD SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS.

By Professor A. Cayley.

TO fix the ideas, the equations are assumed to be without constant terms. Supposing the system to be insufficient for the determination of the ratios of the unknown quantities, then regarding the unknown quantities as having a definite order of arrangement, there are certain solutions which may be regarded as standard solutions. Take the unknown quantities to be A, B, C, D, E, F, G, &c.; then assuming $A = 0$, or else A, B, each $= 0$, or else A, B, C, each $= 0$, as many equations as possible, the system as thus modified will have a definite solution; for instance, the assumed equations A, B, C, D, E, each $= 0$, may give a definite solution in which F is not $= 0$, and it may then for convenience be put $= 1$. We have thus a solution beginning with $F = 1$.

---

* See *Messenger of Mathematics*, vol. x. p. 23, or *Proceedings of the Cambridge Philosophical Society*, vol. iii. p. 383.

This being so, there will be a solution or solutions with $F=0$; we cannot then have A, B, C, D, E, each $=0$, but we again assume $A=0$, or else A, B, each $=0$, as many equations as possible; suppose A, B, C, each $=0$ give a definite solution, with D not $=0$; and then taking it for convenience to be $=1$, we have a solution beginning $D=1$, and for which $F=0$. Going on in this manner we obtain, it may be, a solution beginning $B=1$, and for which $D=0, F=0$; and so on, the process stopping, if not sooner, with a solution beginning $A=1$, and with the initial letters of the preceding solutions, each $=0$. We have in this manner the system of standard solutions, of a form such as

	A	B	C	D	E	F	G	...
=	0	0	0	0	0	1	*	,
=	0	0	0	1	*	0	*	
=	0	1	0	0	*	0	*	
=	1	0	*	0	*	0	*	

where the * denotes a value which is not in general $=0$, but which may in any particular case happen to be so.

For instance, let it be required for the binary quartic $(a, b, c, d, e)(x, y)^4$, to find the asyzygetic seminvariants of the degree 4 and weight 6. Assuming for the seminvariant the value in the left-hand column of the diagram, the unknown coefficients being A, B, C, D, E, F, G, this must be reduced to zero by the operation

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e;$$

and we thus obtain as many equations as there are terms of the degree 4 and weight 5, as appearing by the second column

		$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e$	
A a^2ce	a^2be	$2C + 2A$	$=0,$
B „ d^2	„ cd	$D \quad + 6B + 4A$	$=0,$
C ab^2e	ab^2d	$3F + 2D$	$=0,$
D „ bcd	„ bc^2	$2G + 6E + 3D$	$=0,$
E „ c^3	a^0b^3c	$4G + 3F$	$=0;$
F a^0b^3d			
G „ b^2c^2			

viz. the equations are

$$\begin{array}{|cccccccc|c}
A & B & C & D & E & F & G & \\
\hline
2 & +2 & & & & & & = 0, \\
4+6 & & +1 & & & & & = 0, \\
 & & 4+2 & & +3 & & & = 0, \\
 & & & 3+6 & & +2 & & = 0, \\
 & & & & & & 3+4 & = 0. \\
\hline
\end{array}$$

We have first a solution beginning $B=1$, and secondly a solution beginning $A=1$, with $B=0$: the resulting two seminvariants, say P and Q, are

		$P=$	$Q=$	I	II
A	$a^2c\,e$	0	1	1	1
B	„ d^2	1	0	−1	0
C	$a\,b^2e$	0	−1	−1	−1
D	„ bcd	−6	−4	+2	−4
E	„ c^3	+4	3	−1	+3
F	a^0b^3d	+4	4		+4
G	„ b^2c^2	−3	−3		+3

.

As is known, there is no irreducible solution, but only the composite forms

$$\mathrm{I} = a(ace - ad^2 - b^2e + 2bcd - c^3,$$

$$\mathrm{II} = (ac - b^2)(ae - 4bd + 3c^2),$$

the developed values of which are given above: II (as a form beginning $A=1$ and with $B=0$) can be nothing else than, and is in fact $=Q$: and so I (as a form beginning with $A=1$, $B=-1$) can be nothing else than, and is in fact $=Q-P$; that is, we have

$$P = -\mathrm{I} + \mathrm{II}, \quad \text{or} \quad \mathrm{I} = -P + Q,$$

$$Q = \quad \mathrm{II}, \quad \mathrm{II} = \quad Q;$$

and so in general we have a standard set of values for the asyzygetic seminvariants of a given degree and weight; or, what is the same thing, for the covariants of a given deg.-order.

SUR UNE APPLICATION D'UN DETERMINANT.

Par A. LEGOUX, Professeur à la faculté des sciences à Grenoble.

EN lisant dans le numéro du *Quarterly Journal* du mois de Juin, 1880, la note de Mr. Scott sur une forme spéciale de déterminant découverte par Mr. Glaisher, j'ai pensé que l'on pourrait se servir de cette forme pour résoudre algébriquement et *d'une manière uniforme les équations du* 3^e *et du* 4^e *degré.* Je ne sais si cette méthode est nouvelle; elle est en tout cas une conséquence bien naturelle de la forme remarquable du déterminant de Mr. Glaisher.

I. Considérons le déterminant suivant d'ordre n

$$R = \begin{vmatrix} x & a_1 & a_2 \ldots\ldots a_{n-1} \\ a_{n-1} & x & a_1 \ldots\ldots a_{n-2} \\ a_{n-2} & a_{n-1} & x \quad a_1 \ldots a_{n-3} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1 & a_2 & a_3 \qquad\qquad a_{n-1} \quad x \end{vmatrix}$$

supposons que x soit une inconnue, et a_1, a_2... des nombres donnés, en égalant à 0 on a une équation d'ordre n, ne contenant pas de terme de degré $n-1$ et renfermant $n-1$ quantités connues. Si ω est une racine primitive de l'equation $x^n - 1 = 0$, le déterminant précédent est le produit des facteurs lineaires suivants,

$$\begin{aligned} &x + a_1\omega + a_2\omega^2 + a_3\omega^3 + \ldots a_{n-1}\omega^{n-1}, \\ &x + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots a_{n-1}\omega^{2(n-1)}, \\ &x + a_1\omega^3 + a_2\omega^6 + a_3\omega^9 + \ldots a_{n-1}\omega^{3(n-1)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &x + a_1\omega^n + a_2\omega^{2n} + a_3\omega^{3n} + \ldots a_{n-1}\omega^{n(n-1)}, \end{aligned}$$

le dernier facteur est égal à

$$x + a_1 + a_2 + a_3 + \ldots + a_{n-1}.$$

On voit que l'on obtiendra les racines de l'équation $R = 0$, en égalant chacun de ces polynomes à 0 et que ces racines seront des fonctions linéaires des quantités a_1, a_2... et des racines imaginaires de l'unité.

On voit aussi que si l'on désigne ces racines par les lettres $x_1, x_2 \ldots x_n$, et si l'on représente par le symbole f l'opération qui consiste, étant donnée l'expression $A_1 a_1 + A_2 a_2 \ldots + A_{n-1} a_{n-1}$, à multiplier le premier terme par ω, le seconde par ω^2, le troisième par ω^3 etc., on aura

$$\left.\begin{aligned} -x_1 &= a_1\omega + a_2\omega^2 + a_3\omega^3 + \ldots a_{n-1}\omega^{n-1} \\ -x_2 &= a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots a_{n-1}\omega^{2(n-1)} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots(1).$$

d'où $x_2 = f(x_1)$, de même

$$x_3 = f(x_2),$$
$$x_4 = f(x_3),$$
$$\ldots\ldots\ldots$$

II. Si l'on développait le déterminant R, et si on l'ordonnait relativement aux puissances décroissantes de x, on trouverait en l'égalant à 0 une équation

$$x^n + p_2 x^{n-2} - p_3 x^{n-3} \ldots \pm p_n = 0 \ldots\ldots\ldots\ldots(2),$$

$p_2, p_3 \ldots p_n$ étant des fonctions rationnelles données des quantités $a_1, a_2 \ldots a_{n-1}$.

On peut toujours écrire une équation algébrique sous la forme précédente en faisant disparaître le second terme.

On voit que les coefficients $p_2, p_3 \ldots$ sont en même nombre que les quantités $a_1, a_2 \ldots$, que par conséquent si l'on supposait connus les $p_2, p_3 \ldots$ c'est à dire si l'on partait de l'équation donnée sous la forme (2), on pourrait chercher à l'identifier avec l'équation $R = 0$; ce qui reviendrait à résoudre un système de $n-1$ équations à $n-1$ inconnues pour déterminer les $a_1, a_2 \ldots a_{n-1}$ en fonction des $p_2, p_3 \ldots p_n$. Les quantités $a_1, a_2 \ldots$ étant connues, on aurait les racines de l'équation proposée an moyen des formules (1). Appliquons cette méthode à l'equation du 3e degré:

$$x^3 + p_2 x - p_3 = 0,$$

identifions cette équation avec la suivante

$$\begin{vmatrix} x & a_1 & a_2 \\ a_2 & x & a_1 \\ a_1 & a_2 & x \end{vmatrix} = \left\{\begin{matrix} (x + a_1\omega + a_2\omega^2) \\ (x + a_1\omega^2 + a_2\omega) \\ (x + a_1 + a_2) \end{matrix}\right\} = 0.$$

On trouve

$$3a_1a_2 = -p_2, \quad a_1^3 + a_2^3 = -p_3.$$

On trouve a_1^3 et a_2^3 en résolvant une équation du 2e degré ; on retombe sur les calculs fournis par la méthode bien connue pour déterminer a_1 et a_2. Ces quantités étant déterminées en fonctions de p_2 et p_3 les trois racines de l'équation cubique proposée seront données par les équations linéaires

$$x + a_1 + a_2 = 0, \quad x + a_1\omega + a_2\omega^2 = 0, \quad x + a_1\omega^2 + a_2\omega = 0.$$

Opérons de même sur l'équation du 4e degré

$$x^4 + p_2x^2 - p_3x + p_4 = 0.$$

Identifions là avec la suivante :

$$\begin{vmatrix} x & a_1 & a_2 & a_3 \\ a_3 & x & a_1 & a_2 \\ a_2 & a_3 & x & a_1 \\ a_1 & a_2 & a_3 & x \end{vmatrix} = \begin{Bmatrix} (x + a_1 i - a_2 - a_3 i) \\ (x - a_1 + a_2 - a_3) \\ (x - a_1 i - a_2 + a_3 i) \\ (x + a_1 + a_2 + a_3) \end{Bmatrix} = 0, \; i = \sqrt{(-1)},$$

on trouve

$$-p_2 = 2(a_2^2 + 2a_1a_3),$$

$$p_3 = -4a_2(a_1^2 + a_3^2),$$

$$p_4 = (a_2^2 - 2a_1a_3)^2 - (a_1^2 + a_3^2)^2,$$

substituant dans la troisième les valeurs de a_1a_3 et de $a_1^2 + a_3^2$ données par les deux premières, on a

$$p_4 = \left(a_2^2 + \frac{p_2 + 2a_2^2}{2}\right)^2 - \frac{p_3^2}{16a_2^2}.$$

On trouve ainsi pour déterminer a_2^2 une équation du 3e degré, a_2 étant connu, ou trouvera sans peine a_1 et a_3 ; enfin les quantités a_1, a_2, a_3, étant exprimées en fonction de p_2, p_3, p_4, les racines de l'équation du quatrième degré seront données par les équations linéaires suivantes :

$$x + a_1 i - a_2 - a_3 i = 0, \quad x - a_1 + a_2 - a_3 = 0,$$

$$x - a_1 i - a_2 + a_3 i = 0, \quad x + a_1 + a_2 + a_3 = 0.$$

ON SPHERICAL CYCLOIDAL AND TROCHOIDAL CURVES.

By Henry M. Jeffery.

1. THE spherical epicycloid was first considered by Hermann in the *Acta Petropolitana*, 1726, when this writer wrongly thought that all these curves were rectifiable. In the *Academie des Sciences*, 1736, three memoirs are written by John Bernouilli, Clairaut and Nicole, by the methods of solid geometry on the rectification and quadrature of this epicycloid. Hermann's mistake was pointed out by Bernouilli, and the rectifiable case shown to exist when the rolling circle was a great circle of the sphere.* In the Acts of St. Petersburgh for 1779, the celebrated Lexell, who has the credit of founding Spherical Analytical Geometry, applied spherical coordinates to investigate the rectification, quadrature, and curvature of both epicycloids and epitrochoids. In 1830, Gudermann of Cleves considered the spherical cycloid alone, the spherical catenary and ellipse in his original treatise on Spherics; his exposition is clearer, but less extensive than Lexell's.

It may be worth while to reconsider the subject, and not only to obtain the known results by direct processes, but to add other theorems on these roulettes.

Certain propositions relating to spherical curvature and quadrature, and Lexell's expressions for cycloidal arcs will first be prefixed as lemmas.

2. To express the spherical radius of curvature at a point in a curve in terms of the polar coordinates of the point.

The polar equation to a small circle, whose radius is r, and the coordinates of whose pole are β, α, is

$$\cos r = \cos\beta\cos\rho + \sin\beta\sin\rho\cos(\theta-\alpha),$$

where ρ, θ are the polar coordinates of a current point.

By differentiating with respect to the variables

$$\cot\beta = \cot\rho\cos(\theta-\alpha) - \sin(\theta-\alpha)\frac{d\theta}{d\rho}.$$

* These writers were incited to their task by a problem proposed by Offenburg in the Acta Eruditorum, 1718: Testitudinem hemisphericam fenestris ovalibus perforare, quarum unaquæque peripheriam absolute rectificabilem habeat. The problem was evidently suggested by Viviani's *Florentine Enigma*, but no solution by the proposer is recorded. A sketch of the early history of Analytical Spherics may be read in the *Aperçu Historique* of M. Chasles, pp. 235–240.

Hence $$\frac{\cos r}{\sin\beta} = \operatorname{cosec}\rho\cos(\theta-\alpha) - \cos\rho\sin(\theta-\alpha)\frac{d\theta}{d\rho},$$

and

$$\frac{\cos^2\beta - \cos^2 r}{\sin^2\beta} = -\cos^2(\theta-\alpha) + \sin^2\rho\sin^2(\theta-\alpha)\left(\frac{d\theta}{d\rho}\right)^2,$$

$$\frac{\sin^2 r}{\sin^2\beta} = \sin^2(\theta-\alpha)\left\{1+\sin^2\rho\left(\frac{d\theta}{d\rho}\right)^2\right\} = \sin^2(\theta-\alpha)\left(\frac{ds}{d\rho}\right)^2.$$

This relation between r and β *in plano*, and $\sin\gamma$ and $\sin\beta$ in spherics, may be established geometrically.

If A be the centre of curvature, O the origin of polar coordinates, and P the point of a curve,

$$\frac{\sin r}{\sin\beta} = \frac{\sin AOP}{\sin APO} = \frac{\sin(\theta-\alpha)}{\dfrac{d\rho}{ds}}.$$

Again differentiate for the third point of contact:

$$\sin(\theta-\alpha)\frac{d^2s}{d\rho^2} + \cos(\theta-\alpha)\frac{ds}{d\rho}\frac{d\theta}{d\rho} = 0.$$

The coordinates (α, β) have been thus obtained in terms of the current coordinates of a point on the curve.

For the radius of curvature

$$\cot r\frac{ds}{d\rho} = \operatorname{cosec}\rho\cot(\theta-\alpha) - \cos\rho\frac{d\theta}{d\rho},$$

$$-\cot r\left(\frac{ds}{d\rho}\right)^2\frac{d\theta}{d\rho} = \operatorname{cosec}\rho\frac{d^2s}{d\rho^2} + \cos\rho\frac{ds}{d\rho}\left(\frac{d\theta}{d\rho}\right)^2,$$

or $$-\cot r\left(\frac{ds}{d\rho}\right)^3\frac{d\theta}{d\rho} = \cos\rho\left(\frac{d\theta}{d\rho}\right) + \sin\rho\frac{d\theta}{d\rho}\frac{d^2\theta}{d\rho^2} + \cos\rho\left(\frac{ds}{d\rho}\right)^2\left(\frac{d\theta}{d\rho}\right)^2.$$

Gudermann's expression for the radius (*Sphärik*, § 34) agrees with this.

The corresponding form *in plano* is

$$-\frac{1}{r}\left(\frac{ds}{d\rho}\right)^2\frac{d\theta}{d\rho} = \frac{1}{\rho}\frac{d^2s}{d\rho^2} + \left(\frac{d\theta}{d\rho}\right)^2\frac{ds}{d\rho}.$$

Both Gudermann (§ 31) and Greatheed (*Camb. Math. Journal*, vol. III.) have determined the radius and coordinates of the centre of curvature, when ρ is the independent variable; these formulæ are analogous to the usual expressions *in plano*, but inconvenient for cycloidal curves.

Expressions for curvature are also given by Gudermann (*Sphärik*, § 26) in terms of rectangular coordinates.

3. In cycloidal and trochoidal curves, the radius of curvature and the evolutes are most conveniently found by the relation between the perpendicular and radius vector drawn from the pole.

Let p, r be the coordinates of the curve, p', r' those of its evolute, ρ, ρ' their radii of curvature.

If $p=f(r)$ be the equation to the curve, then, as in planimetry,

$$\cos r' = \cos r \cos\rho + \sin p \sin\rho, \text{ since } \frac{\sin p}{\sin r} = \sin r \frac{d\theta}{ds},$$

$$\sin^2 p' = \sin^2 r - \sin^2 p.$$

In differentiating, ρ, r' remain constant, while r, p vary,

$$\tan\rho = \frac{\sin r}{\cos p}\frac{dr}{dp} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

By eliminating from these four equations, the equation to the evolute is found.

Cor. 1. In planimetry,

$$\frac{\rho'}{p'} = \frac{d\rho}{dp}; \quad \text{in spherics, } \frac{\tan\rho'}{\sin p'} = \frac{d(\sin\rho)}{d(\sin p)}.$$

Cor. 2. The following expressions for the length of a curve, and of the area inclosed by it, are sometimes useful.

$$s = \int dr \sin r (\sin^2 r - \sin^2 p)^{-\frac{1}{2}}.$$

$$\text{Area} = \iint \sin r\, dr\, d\theta = \int dr (\cos a - \cos r) \operatorname{cosec} r (\sin^2 r - \sin^2 p)^{-\frac{1}{2}} \sin p.$$

4. An analogous expression may be hence deduced to the plane formula for the radius of curvature in terms of the polar tangential coordinates

$$\rho = p + \frac{d^2 p}{d\theta^2}.$$

In vectorial polar coordinates, as appears from the geometry,

$$\left(\frac{dr}{\sin r d\theta}\right)^2 = \frac{\sin^2 r}{\sin^2 p} - 1, \text{ or } \left(\frac{d}{d\theta}\cot r\right)^2 + \cot^2 r = \cot^2 p \ldots (B).$$

(The formulæ (A) and (B) were proposed in the Mathematical Tripos Examination, Jan. 17, 1882.)

By polarising, when $r = \frac{1}{2}\pi - p'$, $p = \frac{1}{2}\pi - r'$,

$$\left(\frac{d}{d\theta}\tan p\right)^2 + \tan^2 p = \tan^2 r.$$

By differentiation, the formula (A) becomes

$$\tan\rho = \frac{\sin r}{\cos p}\frac{dr}{dp} = \frac{\cos^3 r}{\cos^3 p}\left\{\tan p + \frac{d^2}{d\theta^2}(\tan p)\right\}.$$

Ex. 1. The spherical ellipse

$$\tan^2 p = a^2\cos^2\theta + b^2\sin^2\theta,$$

so that $$\tan^2 r\tan^2 p = a^4\cos^2\theta + b^4\sin^2\theta,$$

and $$\tan^2 p\,(a^2 + b^2 - \tan^2 r) = a^2b^2,\ \tan\rho = \frac{a^2b^2\cos^3 r}{\sin^3 p}.$$

Ex. 2. The rhumb-line or equiangular spherical spiral

$$\sin p = m\sin r,\ \tan\rho = \frac{1}{m}\tan r.$$

The coordinates of the evolute are thus related:

$$m^2\tan^2 r' = (1 - m^2)\sin^2 r = \sin^2 p'.$$

The evolute is not a similar spiral, as *in plano*.

5. There is a close analogy in planimetry and spherics for the expressions for the areas of the spaces inclosed by the epicycloid and the two circles, fixed and generating (fig. 1).

Clairaut's method of integrating will be first applied to planimetry. Let O be the centre of the fixed circle, C, C' two successive positions of the centre of the rolling circle, AQP the epicycloid. The area is sought of the space APQ.

Let Qq be drawn concentric with AP; the elemental area $pPQR$ is sought, or, as well, of $pPQq$. It is plain that there can be as many spaces such as $pPQq$ in the circular zone $\{\pi(OQ^2 - OP^2)\}$, as Pp is contained in the circumference of the fixed circle APp. Hence the element

$$pPQR = \tfrac{1}{2}d\theta\,(r^2 - a^2),$$

if r, a, b denote OQ, OP, CP; θ, ϕ, the angles AOP, POQ; and since

$$r^2 = (a + b)^2 + b^2 - 2b\,(a + b)\cos\phi,\ \text{and}\ a\theta = b\phi,$$

the area of APQ

$$= \frac{b^2}{a}(a+b)\int(1-\cos\phi)\,d\phi$$

$$= \frac{b^2}{a}(a+b)(\phi-\sin\phi),$$

when $\phi=\pi$, for a half-roll of the generating circle,

$$\text{area} = \frac{\pi b^2}{a}(a+b).$$

This agrees with the known value of half the area of the epicycloid limited by the fixed circle $\frac{\pi b^2}{2a}(3a+2b)$, differing from the former by half the area of the generating circle.

Cor. When $b=\infty$, the epicycloid becomes the involute of the generating circle. The area inclosed by the involute, its radius of curvature, and the fixed circle or evolute, is equal to the area of the involute bounded by the radius vector and the initial line or vector. Hymer's *Int. Calc.* p. 201. This result is derived by aid of the equation $p^2+a^2=r^2$,

$$\int\tfrac{1}{2}(r^2-a^2)\,d\theta = \tfrac{1}{2}\int p^2 d\theta = \tfrac{1}{2}\int a^2\theta^2 d\theta = \tfrac{1}{2}\int r^2 d\phi = \tfrac{1}{6}p^3.$$

It appears from the convertible expressions

$$r^2 = p^2 + \left(\frac{dp}{d\theta}\right)^2, \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2,$$

that

$$r^2 d\phi = p^2 d\theta.$$

6. Lexell's method of determining the elemental arc of a roulette is shown in the cases of the plane cycloid and epicycloid.

In fig. 1 to the increment QR of the epicycloid there corresponds the increment SQ or Rq of the generating circle, and RS or Qq is concentric with or parallel to the fixed circle APp.

The ratio of these increments is $\frac{a+b}{ab}PQ$.

If the infinitesimal parallelogram $QSRq$ be considered,

$$QR : qR :: \sin QqR : \sin qQR :: \sin SQq : \sin qQR$$
$$:: \cos CQq : \cos PQq :: \sin OQC : \sin OQP$$
$$:: OC.PQ : OP.QC :: (a+b)PQ : ab,$$

so that $QR = \frac{(a+b)}{a}PQ\,d\phi$, since $qR = bd\phi$, and $PCQ = \phi$.

$$\text{The arc of the epicycloid} = \int \frac{a+b}{a}\, 2b \sin \tfrac{1}{2}\phi\, d\phi$$

$$= \frac{4b}{a}(a+b)(1-\cos\tfrac{1}{2}\phi),$$

reckoned from an apse.

Cor. In the cycloid, $ds = PQ\,d\phi$, and $s = 4b\,(1-\cos\tfrac{1}{2}\phi)$.

7. To obtain the equation to the spherical cycloid.

(Companion-figures should be drawn to figs. 1, 2, of an hypocycloid, in which the generating circle PQS should roll inside the fixed circle. The circles are supposed to be drawn on a sphere.)

Lexell's method gives the arc in a form analogous to its plane congener. Let r, p denote the radius vector of a moving point and perpendicular from the pole on a tangent thereat; a, b, the spherical radii of the fixed and rolling circles

$$QR : qR :: \sin QqR : \sin qQR :: \sin OQC : OQP$$

$$:: \sin(a+b)\sin PQ : \sin a \sin b.$$

Hence $ds = \cos b \sin PQ\, d\phi$, since $a = \frac{1}{2}\pi$ in the case of the spherical cycloid.

Also from the spherical triangles OCQ, OPQ,

$$\cos r = \cot b\,(1-\cos PQ) = \sin b \cos b\,(1-\cos\phi),$$

$$\sin\tfrac{1}{2}PQ = \sin b \sin\tfrac{1}{2}\phi.$$

$$1 - \frac{\sin^2 p}{\sin^2 r} = \left(\frac{dr}{ds}\right)^2 = \left(\frac{\sin b \sin\phi}{\sin r \sin PQ}\right)^2,$$

$$\cos^2 p - \cos^2 r = \cos^2\tfrac{1}{2}\phi \sec^2\tfrac{1}{2}(PQ)$$

$$= \frac{\sin 2b - \cos r}{\sin 2b - \sin^2 b \cos r}.$$

8. The same relation may be otherwise obtained.

Since PQ and p are both perpendicular to the tangent at Q, they will meet if produced, in the pole of QR. In the exterior triangle thus formed

$$\sin p = \cos PQ \cos CPQ = \cos PQ \tan\tfrac{1}{2}(PQ)\cot b.$$

Also $\sin\tfrac{1}{2}(PQ) = \sin b \sin\tfrac{1}{2}\phi$, $\cos PQ = 1 - \cos r \tan b$.

Hence

$$\sin^2 p = \frac{\cos^2 PQ}{1+\cos PQ}\cos r\cot b = \frac{\cos r\cot^2 b}{2\cot b - \cos r} - \cos^2 r,$$

$$\cos^2 p - \cos^2 r = \frac{\sin 2b - \cos r}{\sin 2b - \sin^2 b\cos r} = \sin^2 CQP \text{ (see } \textit{infrà}, \S\,14).$$

By a third method Gudermann (Sphärik, §§ 39, 40) obtains the differential equation to the cycloid,

$$\tan r\frac{d\theta}{dr} = \frac{\cos b - \sin b\cos r}{\sqrt{(\sin 2b\cos r - \cos^2 r)}},$$

which is readily deduced from the former value of $\frac{ds}{dr}$.

9. To determine the length of the spherical cycloid.

For integration, it is convenient to select for the independent variable the perpendicular from C on QP the normal. Call $CN = z$.

Then $\cos\frac{1}{2}\phi = \tan z\cot b$, $\cos b = \cos\frac{1}{2}(PQ)\cos z$,

$$\sin\tfrac{1}{2}(PQ) = \sin\tfrac{1}{2}\phi\sin b.$$

Lexell's expression for

$$ds = \cos b\sin PQ\,d\phi = \frac{\sin 2b\sin\frac{1}{2}\phi}{\cos z}\,d\phi$$
$$= -4\frac{\cos^3 b}{\cos^3 z}\,dz.$$

Hence $\quad \frac{1}{2}s\sec^3 b = C - \dfrac{\sin z}{\cos^2 z} - \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}z).$

The arc begins when $z = b$, so that the semi-arc of the cycloid

$$= \frac{\sin b}{\cos^2 b} + \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}b).$$

10. On the quadrature of the spherical cycloid.

Two different areas are here considered, (1) AQP bounded by epicycloid and the two circles, (2) that which is bounded by AQ, OQ produced, and the fixed great circle APp.

(1) The elemental area of the space $pPQq$ comprised between two consecutive positions of the generating circle

$$= \cos OQ\,d(AOP).$$

For there are as many spaces, such as $pPQq$, in the spherical zone, bounded by circles whose angular radii are OQ and OA, as Pp is contained in the whole circumference of the great circle APp.

And it is known that the area of this zone is the product of the height or distance between the planes OAP, OQq, by the circumference of a great circle. Such is Clairaut's reasoning.

In the sequel

$$\theta = AOP,\ \phi = PCQ,\ \chi = AOQ,\ r = OQ,$$
$$CP = CQ = b : OP = \tfrac{1}{2}\pi.$$

The whole area

$$AQP = \int_0^{\phi} \cos r\, d\theta = \sin^2 b \cos b \int (1 - \cos\phi)\, d\phi$$
$$= \sin^2 b \cos b\, (\phi - \sin\phi).$$

The summation commences from the cusp A; and for half a roll the area swept out is $\pi \sin^2 b \cos b$.

(2) The required area is $= \int_r^{\frac{1}{2}\pi} \int_0^{\chi} \sin r\, dr\, d\chi$

$$= \int_{\chi}^{0} \cos r\, d\chi,$$

where χ is the vector angle AOQ.

Next make PCQ or ϕ the independent variable.

From § 7, $\quad \cos r = \sin b \cos b\, (1 - \cos\phi),$

$$\sin r dr = -\sin b \cos b \sin\phi d\phi.$$

Also from the differential equation to the cycloid in § 8,

$$\tan r \frac{d\chi}{dr} = \frac{\cos b - \sin b \cos r}{\sin b \cos b \sin\phi} = -\operatorname{cosec} r\, (\cos b - \sin b \cos r) \frac{d\phi}{dr}.$$

The required area

$$= \int_0^{\phi} d\phi \frac{\cos^2 r}{1 - \cos^2 r} (\cos b - \sin b \cos r)$$

$$= \int_0^{\phi} d\phi \left\{ \frac{\cos b - \sin b \cos r}{1 - \cos^2 r} - \cos b + \sin b \cos r \right\}.$$

Let $u = OQC$, so that

$$\sin u = \frac{\cos b \sin \phi}{\sin r} = -\operatorname{cosec} b \frac{dr}{d\phi};$$

also
$$\cos u = \frac{\sin^2 b + \cos^2 b \cos \phi}{\sin r},$$

$$\sin b \cos u = \frac{\sin b - \cos b \cos r}{\sin r}.$$

The required area

$$= \int_0^\phi d\phi \left\{ \frac{d}{dr} \left(\frac{\sin b - \cos b \cos r}{\sin r} \right) - \cos b + \sin b \cos r \right\}$$

$$= \int_0^\phi d\phi \left\{ \frac{d}{dr} (\sin b \cos u) - \cos b + \sin b \cos r \right\}$$

$$= \int_0^\phi d\phi \left\{ -\operatorname{cosec} u \frac{d}{d\phi} \cos u - \cos^3 b - \sin^2 b \cos b \cos \phi \right\}$$

$$= \sin^{-1} \left(\frac{\cos b \sin \phi}{\sin c} \right) - \phi \cos^3 b - \sin^2 b \cos b \sin \phi$$

$$= OQC - \phi \cos^3 b - \sin^2 b \cos b \sin \phi.$$

Gudermann (*Sphärik*, p. 152).

For a half-roll, $\phi = \pi$, and half the area of the cycloid

$$= \pi (1 - \cos^3 b).$$

The difference between the areas (1) and (2) is half the spherical surface cut off by the generating circle, $\pi(1 - \cos b)$, as it should be.

11. To find the tangential equation to the spherical cycloid.

If the relation between p and r is known for a spherical curve, the polar equation, both vectorial and polar, is readily deduced.

As in § 4

$$\sec^2 r = \sec^2 p + \frac{d}{d\phi} (\tan p)^2 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where ϕ is the inclinaiion of p to the initial line.

Also from § 7 the relation between p and r is

$$\sin 2b \sin^2 p \sec^3 r - (1 - \sin^2 b \cos^2 p) \sec^2 r + \sin 2b \sec r - \sin^2 b = 0 \ldots (2).$$

The elimination of $\sec r$ from (1) and (2) determines the differential tangential equation.

12. To find the spherical radius of curvature at any point of a spherical cycloid.

The formula in polar coordinates for their radius is given in § 2,

$$-\cot R\left(\frac{ds}{dr}\right)^3 = \cos r\left(\frac{ds}{dr}\right)^2 \frac{d\theta}{dr} + \frac{d}{dr}\left(\sin r \frac{d\theta}{dr}\right).$$

The differential equation to the cycloid (§ 8) is

$$\sin r \frac{d\theta}{dr} = \frac{\cos r\,(\cos b - \sin b \cos r)}{\sqrt{\{(\sin 2b - \cos r)\cos r\}}},$$

whence $$\left(\frac{ds}{dr}\right)^2 = 1 + \sin^2 r\left(\frac{d\theta}{dr}\right)^2 = \sin^2 r \frac{\sin 2b - \sin^2 b \cos r}{\sin 2b - \cos r}$$

$$-\frac{d}{dr}\left(\sin r \frac{d\theta}{dr}\right) = \sin b \sin r \cos r\,(\cos^2 b + \cos^2 r - 3 \sin b \cos b \cos r)$$
$$\div (\sin 2b \cos r - \cos^2 r)^{\frac{3}{2}}.$$

After substitution,

$$-\cot R\,(\sin b \cos r)^{\frac{1}{2}}(2\cos b - \sin b \cos r)^{\frac{3}{2}}$$
$$= \cos^2 b - 3 \sin b \cos b \cos r + \sin^2 b \cos^2 r.$$

Gudermann has elegantly expressed this relation in terms of the chord PQ, and thereby shewn its analogy to the plane property.

As in § 7, $\quad 1 - \cos PQ = \tan b \cos r,$

hence $\quad \tan R\,(1 - \cos PQ - \cos^2 PQ) = \sin PQ\,(1 + \cos PQ),$

$$\sin R \sqrt{(1 + \sin^2 PQ)} = \sin PQ\,(1 + \cos PQ).$$

13. The same value may be more simply obtained from the formula of § 3 for the radius of curvature

$$\tan R = \frac{\sin r}{\cos p}\frac{dr}{dp}.$$

As in § 8, $\quad \sin p = \cos PQ \tan \tfrac{1}{2} PQ \cot b,$

$$\cos r = \cot b\,(1 - \cos PQ).$$

Hence, by differentiating,

$$\cot R = \tan \tfrac{1}{2} PQ - \tfrac{1}{2} \cot PQ \sec^2 \tfrac{1}{2} PQ$$
$$= \frac{\sin PQ - \cot PQ}{1 + \cos PQ} = \frac{1 - \cos PQ - \cos^2 PQ}{\sin PQ\,(1 + \cos PQ)}.$$

14. To find the evolute of the spherical cycloid.
The equations in terms of p' and r' are established in § 3,

$$\cos r' = \cos r \cos R + \sin p \sin R,$$

$$\sin^2 p' = \sin^2 r - \sin^2 p.$$

Substitute the values of r, p, and R, obtained from §§ 12, 13,

$$\cos r' \sqrt{(1 + \sin^2 PQ)} = 2 \cot b \sin^2 \tfrac{1}{2} PQ.$$

Again, since OP is $\frac{1}{2}\pi$,

$$p' = CPQ, \quad \cos p' = \cot b \tan \tfrac{1}{2} PQ.$$

By eliminating PQ, the equation to the evolute is found

$$\cos r' \sqrt{(1 + 6 \cos^2 p' \tan^2 b + \cos^4 p' \tan^4 b)} = 2 \cos^2 p' \tan b.$$

An apse and cusp in the cycloid correspond to a cusp and apse in its evolute.

If $PQ = 2b$, $p' = 0$, $\tan r' = \operatorname{cosec} 2b$, $R' = 0$.

If $PQ = 0$, $r' = p' = \frac{1}{2}\pi$, $\tan R' = 4 \tan b$.

15. To obtain the equation to the spherical epicycloid.
The differential equation has been given by Lexell, as in § 7,

$$ds = d\phi \sin (a + b) \operatorname{cosec} a \sin PQ.$$

Also from the triangles OCQ, PCQ, fig. (2),

$$\cos r = \cos (a + b) \cos b + \sin (a + b) \sin b \cos \phi,$$

$$\sin \tfrac{1}{2} PQ = \sin \tfrac{1}{2} \phi \sin b.$$

Hence, by differentiation,

$$\frac{dr}{ds} = \sin a \cos \tfrac{1}{2} \phi \operatorname{cosec} r \sec \tfrac{1}{2} PQ.$$

Also $$\sin^2 r - \sin^2 p = \sin^2 r \left(\frac{dr}{ds}\right)^2$$

$$= \frac{1}{2 - \sin^2 b \,(1 - \cos \phi)} (1 + \cos \phi) \sin^2 a,$$

$$\cos^2 p - \cos^2 r = \frac{\sin^2 a}{\sin^2 b} \frac{\cos r - \cos (a + 2b)}{\cos r + 2 \sin a \cot b + \cos a}.$$

This equation may be obtained briefly from the figure 1.

As in § 8, let p, PQ be produced to meet in the pole of QR.

In the triangle thus formed,

$$\sin p = \cos a \sin PQ + \sin a \cos PQ \cos CPQ$$

$$= \tan \tfrac{1}{2} PQ \{\cos a (1 + \cos PQ) + \sin a \cot b \cos PQ\}$$

$$\sin^2 p = \frac{1 - \cos PQ}{1 + \cos PQ} \{\cos r + \cos a + \sin a \cot b\}^2.$$

Since $(1 - \cos PQ) \sin (a + b) = (\cos a - \cos r) \sin b$,

$$\sin^2 p (\cos r + 2 \sin a \cot b + \cos a)$$

$$= (\cos a - \cos r) (\cos r + \cos a + \sin a \cot b)^2.$$

This can be reduced to the former expression.

16. To determine the length of the spherical epicycloid.

The process adopted for the cycloid in § 9 may be applied generally

$$\sin a \operatorname{cosec} (a + b)\, ds = \sin PQ\, d\phi = -4 \cos^2 b \sec^3 z\, dz.$$

When $z = b$, the arc vanishes,

$$\tfrac{1}{2} s \sin a \operatorname{cosec} (a + b) \sec^2 b = \tan b \sec b - \tan z \sec z$$

$$+ \log \{\tan \tfrac{1}{4}\pi + \tfrac{1}{2} b) \cot (\tfrac{1}{4}\pi + \tfrac{1}{2} z).$$

Cor. The arc is rectifiable only, when b is $\frac{1}{2}\pi$, or $\cos b = 0$, and $PQ = \phi$,

$$ds = d\phi \sin \phi \cot a, \quad s = 2 \cot a \sin^2 \tfrac{1}{2}\phi.$$

17. On the quadrature of the spherical epicycloid (fig. 2).

For the limits taken in each case, see § 10.

Let $OQ\,(= r)$, $QOA\,(= \chi)$, be the polar coordinates of Q; $OCQ = \phi$, $POA = \theta$ as before.

The following relations subsist:

$$\cos r = \cos (a + b) \cos b + \sin (a + b) \sin b \cos \phi$$

$$= \cos a - \sin (a + b) \sin b (1 - \cos \phi)$$

$$= \cos (a + 2b) + \sin (a + b) \sin b (1 + \cos \phi),$$

$$\cos (\theta - \chi) \sin (a + b) = \cos b \operatorname{cosec} r - \cot r \cos (a + b).$$

By differentiating

$$\sin r\, dr = \sin(a+b)\sin b \sin\phi\, d\phi,$$

$$\sin(\theta-\chi)\sin(a+b)\left(\frac{d\theta}{dr}-\frac{d\chi}{dr}\right)=\{\cos b\cos r-\cos(a+b)\}\operatorname{cosec}^2 r.$$

Hence $$\frac{d\theta}{d\phi}-\frac{d\chi}{d\phi}=\{\cos b\cos r-\cos(a+b)\}\operatorname{cosec}^2 r,$$

and, since $$\theta\sin a=\phi\sin b,$$

$$\frac{d\chi}{d\phi}=\sin b\operatorname{cosec} a-\{\cos b\cos r-\cos(a+b)\}\operatorname{cosec}^2 r,$$

and

$$\sin a\sin^2 r\frac{d\chi}{d\phi}=(\cos a-\cos r)\{\sin(a+b)+\sin b\cos r\}.$$

(1) The area bounded by the radius vector and fixed circle

$$AOQ=\int_a^r\int_0^\chi \sin r\, dr\, d\chi=\int_0^\chi d\chi(\cos a-\cos r)$$

$$=\int_0^\phi d\phi\,\{(\cos a-\cos r)\sin b\operatorname{cosec} a-\cos b\}$$

$$+\tfrac{1}{2}\{\cos b-\cos(a+b)\}(1-\cos a)\int_0^\phi\frac{d\phi}{1-\cos r}$$

$$+\tfrac{1}{2}\{\cos b+\cos(a+b)\}(1+\cos a)\int_0^\phi\frac{d\phi}{1+\cos r}$$

$$=-\phi(\cos^3 b-\cot a\sin^3 b)-\sin\phi\sin^2 b(\cos b+\cot a\sin b)$$

$$+(1-\cos a)\tan^{-1}\{(\cos b+\cot\tfrac{1}{2}a\sin b)\tan\tfrac{1}{2}\phi\}$$

$$+(1+\cos a)\tan^{-1}\{(\cos b-\tan\tfrac{1}{2}a\sin b)\tan\tfrac{1}{2}\phi\}$$

$$=-\phi(\cos^3 b-\cot a\sin^3 b)-\sin\phi\sin^2 b(\cos b+\cot a\sin b)$$

$$+OQC-COQ\cos a.$$

For $$\sin r\sin OQC=\sin(a+b)\sin\phi,$$

$$\sin r\sin COQ=\sin b\sin\phi,$$

and $$\cos r=\cos(a+b)\cos b+\sin(a+b)\sin b\cos\phi.$$

For a half-roll, $\phi=\pi$, and half the area of the epicycloid is

$$\pi(1-\cos^3 b+\cot a\sin^3 b).$$

(2) The area of the space bounded by the epicycloid, and the fixed and generating circles $= \iint \sin r \, d\theta \, dr$ (where $AOP = \theta$, as was explained in § 5)

$= \int (\cos a - \cos r) \, d\theta = \sin^2 b \sin(a+b) \operatorname{cosec} a \int (1 - \cos\phi) \, d\phi$

$= \sin^2 b (\cos b + \cot a \sin b)(\phi - \sin\phi)$.

For a half-roll $\phi = \pi$, and the area is

$$\sin^2 b (\cos b + \cot a \sin b) \pi.$$

The difference of the areas in (1) and (2) is half the spherical surface intercepted by the generating circle, viz. $\pi(1 - \cos b)$, as it should be.

18. The differential polar tangential equation to the epicycloid is obtained as in § 11, by eliminating $\sec r$ from a quadratic and cubic equation.

19. To find the radius of curvature of the spherical epicycloid.

From §§ 15, 3, if R denote the radius of curvature,

$$\cos r = \cos a \cos PQ + \sin a \sin PQ \cos OPQ$$
$$= \cos a \cos PQ - \sin a \sin PQ \tan \tfrac{1}{2} PQ \cot b$$
$$\sin p = \cos a \sin PQ + \sin a \cot b \cos PQ \tan \tfrac{1}{2} PQ,$$

$$\cot R = \frac{\cos p}{\sin r} \frac{dp}{dr} = \frac{\cot a \cot PQ + \tfrac{1}{2} \cot b \sec^2 \tfrac{1}{2} PQ (\cot PQ - \sin PQ)}{\cot a + \cot b},$$

$$\operatorname{cosec}^2 R \sin^2 PQ (\cot a + \cot b)^2$$
$$= (\cot a + \tfrac{1}{2} \cot b \sec^2 \tfrac{1}{2} PQ)^2 + \cot^2 b \tan^2 \tfrac{1}{2} PQ.$$

20. To find the evolute of the spherical epicycloid

$$\sin^2 p' = \sin^2 r - \sin^2 p = \sin^2 a (1 - \cot^2 b \tan^2 \tfrac{1}{2} PQ)$$
$$= \sin^2 a \sin^2 CPQ,$$

as appears at once from the figure; whence $\cos PQ$ is known in terms of $\sin^2 p'$.

By § 3, $\cos r' = \cos r \cos R + \sin p \sin R$,

$\cos r' \operatorname{cosec} PQ \operatorname{cosec} a \operatorname{cosec} R (\cot a + \cot b)$

$= (\cot PQ \cot a - \cot b \tan \tfrac{1}{2} PQ)$

$\times \{\cot a \cot PQ + \tfrac{1}{2} \cot b \sec^2 \tfrac{1}{2} PQ (\cot PQ - \sin PQ)\}$

$+ (\cot a + \cot b \cot PQ \tan \tfrac{1}{2} PQ)(\cot a + \cot b)$

$= \cot^2 a \operatorname{cosec}^2 PQ + \tfrac{1}{2} \cot a \cot b \operatorname{cosec}^2 PQ \sec^2 \tfrac{1}{2} PQ$

$+ \tfrac{1}{4} \cot^2 b \sec^4 \tfrac{1}{2} PQ.$

That is,

$$\cos r' \operatorname{cosec} a \sqrt{\left\{\cot^2 a + 2 \cot a \cot b \frac{1}{1+\cos PQ} + \cot^2 b \frac{1+\sin^2 PQ}{(1+\cos PQ)^2}\right\}}$$

$$= \cot^2 a + \cot a \cot b \frac{1}{1+\cos PQ} + \cot^2 b \frac{1-\cos PQ}{1+\cos PQ}.$$

The elimination of $\cos PQ$ will determine the equation to the evolute.

21. On the rectifiable spherical epicycloid (fig. 3).

This case occurs when a great circle rolls on a small circle. As was shewn in § 16, its length is $\cot a(1-\cos\phi)$, and is a multiple of the abscissa of the great circle PQ measured in its plane from P, or of the small circle ASQ measured in its plane. It has not been noticed, that part of this curve is the involute of a small circle, just as *in plano* part of the involute of a circle is a terminal form of an epicycloid.

The following is a direct proof of this property. From § 15, its equation is

$$\cos^2 p - \cos^2 r = \sin^2 a.$$

By §§ 3, 19,

$$\cos R = \cos r \sec a, \quad \text{or} \quad R = \phi = QP, \; r' = p' = a.$$

The spherical radius of curvature is an arc of the generating circle, and touches the fixed circle AQ.

22. On the quadrature of areas enclosed by the rectifiable spherical epicycloid.

As in § 17, the two spaces are selected, which are bounded by the epicycloid and (1) by both the fixed and moving circles, and (2) by the fixed circle and a vector arc from the centre.

(1) $\int(\cos a - \cos r)\, d\theta = \cot a \int(1-\cos\phi)\, d\phi = \cot a\,(\phi - \sin\phi)$,

Clairaut remarks, that this area is the difference of a circular space and a quadrable quantity, so that if from the area in question a spherical sector be substracted, with ϕ for its vertical angle, the remainder would be strictly quadrable. All the spherical epicycloids and epitrochoids (§ 28) would thus also solve the problem of M. Viviani, in which there were to be found on the sphere quadrable geometrical spaces, on the occasion of which problem, M. Offenburg had proposed the additional problem of rectifiable arcs.

(2) To find the second area, the method and notation of § 17 are adopted.

The area required

$$= \int_a^r \int_0^\chi \sin r \, dr \, d\chi = \int_0^\chi d\chi \, (\cos a - \cos r)$$

$$= \operatorname{cosec} a \int dr \, (\cos a - \cos r) \operatorname{cosec}^2 r \sin^2 p$$

$$= \cos^3 a \operatorname{cosec} a \int d\phi \, \frac{\sin^2 \phi \, (1 - \cos \phi)}{1 - \cos^2 a \cos^2 \phi},$$

(since $\cos r = \cos a \cos \phi$, $\sin p = \cos a \sin \phi$,

and, as in § 17, $\sin a \sin^2 r \frac{d\chi}{d\phi} = \cos^2 a - \cos^2 r = \sin^2 p$).

$$= \cot a \int d\phi \, (1 - \cos \phi) \left\{ 1 - \frac{\sin^2 a}{1 - \cos^2 a \cos^2 \phi} \right\}$$

$$= \cot a \int d\phi \, 1 - \cos \phi) - \sin a \cos a \int d\phi \, \frac{\sec^2 \phi}{\sin^2 a + \tan^2 \phi}$$

$$+ \int d\phi \, \frac{\cot a \cot \phi}{1 + \cot^2 a \sin^2 \phi}$$

$$= \cot a \, (\phi - \sin \phi) - \cos a \tan^{-1} (\tan \phi \operatorname{cosec} a) + \tan^{-1} (\sin \phi \cot a)$$

$$= QOA \cos a - \cot QPO - QOP \cos a + \tfrac{1}{2}\pi - QPO$$

$$= AOP \cos a - \tan OPZ + OPZ$$

$$= \cot a \cos^{-1} (\cos r \sec a) - \cos a \cos^{-1} (\tan a \cot r)$$

$$- \sin p \operatorname{cosec} a + \sin^{-1} (\sin p \operatorname{cosec} r).$$

The whole area included by the epicycloid, the radius vector and initial line

$$= AOP + OPZ - \tan OPZ.$$

Hence, the area of the space PAZ bounded by the tangent, the curve, and the initial line

$$= PZO - \pi + \tan OPZ.$$

If the sector $\pi - PZO$ be added, this sum is also strictly quadrable.

23. The involute of a small circle is its own polar curve.

This conclusion is derived from the convertible form of its equation

$$\sin^2 r - \sin^2 p = \sin^2 a.$$

If $\frac{1}{2}\pi - p'$, $\frac{1}{2}\pi - r'$, be written for r, p, the centre being the pole, the polar curve is seen to be identical,

$$\sin^2 r' - \sin^2 p' = \sin^2 a.$$

The order- and class-equations are identical; for by § 4 their differential equations are convertible for this curve.

$$\left(\frac{d}{d\chi}\cot r\right)^2 + \cot^2 r = \cot^2 p = \frac{\sin^2 a + (1+\sin^2 a)\cot^2 r}{\cos^2 a - \sin^2 a \cot^2 r},$$

$$\left(\frac{d}{d\omega}\tan p\right)^2 + \tan^2 p = \tan^2 r = \frac{\sin^2 a + (1+\sin^2 a)\tan^2 p}{\cos^2 a - \sin^2 a \tan^2 p}.$$

If $\sin a = \frac{1}{2}$ in the simplest case, this epicycloid is a cubic in order and class.

By using the notation of § 17,

$$\cos a \cos\phi = \cos r, \quad \sin\phi = \sin(\theta - \chi)\sin r, \quad \phi = \theta \sin a = \tfrac{1}{2}\theta,$$

$$\operatorname{cosec} r = 2\cos\chi \cos\tfrac{1}{2}\theta - \sin\chi\cos\theta \operatorname{cosec}\tfrac{1}{2}\theta,$$

$$(\operatorname{cosec} r - 2\cos\chi\cos r \sec a)^2 = \sin^2\chi \sec^2 a \frac{(\cos^2 a - 2\cos^2 r)^2}{\cos^2 a - \cos^2 r}.$$

Since in Gudermann's rectangular system,

$$x : y : 1 :: \cos\chi : \sin\chi : \cot r,$$

this equation is seen to be a sextic ($\cos a = \frac{1}{3}\sqrt{3}$),

$$3(1 + y^2 + x^2 - \tfrac{4}{3} + \sqrt{3}\,x)^2 (3x^2 + 3y^2 - 1) = y^2 (3x^2 + 3y^2 - 5)^2.$$

The dual form expresses the same involute or epicycloid by implication.

$$3(1 + \eta^2 + \xi^2 - \tfrac{4}{3}\sqrt{3}.\xi)^2 (3\xi^2 + 3\eta^2 - 1) = \eta^2 (3\xi^2 + 3\eta^2 - 5)^2.$$

This dual form is the eliminant of the equations

$$\cos a \sin\phi = \sin p, \quad \cos\phi = \sin(\theta - \omega), \quad \phi = \theta \sin a = \tfrac{1}{2}\theta,$$

where p, ω are the polar tangential coordinates of a tangent to this epicycloid.

In (fig. 3) this involute and in (fig. 3*) both parts of the sextic are drawn.

The preceding sextic may be resolved into two cubics

$$y^2 (x - \sqrt{3}) + (x - \tfrac{1}{3}\sqrt{3})^3 = 0,$$

$$y^2 (x - \tfrac{5}{3}\sqrt{3}) + (x + \tfrac{1}{3}\sqrt{3})(x - \sqrt{3})^2 = 0.$$

The former is its own polar curve, as appears from its tangential equivalent

$$\eta^2 (\xi - \sqrt{3}) + (\xi - \tfrac{1}{3}\sqrt{3})^3 = 0.$$

By writing this cubic in the form

$$(\xi^2 + \eta^2)(1 - \tfrac{1}{3}\sqrt{3}\,\xi) + \tfrac{1}{3}(1 - 3\sqrt{3}\,\xi) = 0,$$

it is recognised to have a double focus at the origin or centre of the fixed circle, and a single focus at the distance $\tan^{-1}(3\sqrt{3})$ from that centre, and, by parity of reasoning, a double cyclic arc.

The second cubic is nodal and of the fourth class, and is foreign to the epicycloid; but it gives a trace of its plane congener, which is generated when a circle rolls on another of half its diameter.

24. This case has been considered by John Bernouilli, whose words are subjoined: "Conceive in the sphere a tropic 60° distant from the Equator, and make the Ecliptic turn on it; a point in the circumference will describe the epicycloid, whose length shall have the ratio $2 : \sqrt{3}$ to the length of its arc of projection; on this hypothesis the ecliptic will be double the tropic, and consequently it must roll upon it twice, before the describing point returns to the point of departure; and the length of the semi-epicycloid will be double of the greatest height or the distance between the tropics; and therefore the length of the whole will be four times their distances."

As in § 16 Cor. the semi-epicycloid $= \cot a\,(1-\cos\phi) = 2\sqrt{3}$, when $\phi = \pi$, $a = \frac{1}{6}\pi$.

"This curve (see fig. 3 of § 23), as it is easy to see from the manner in which it is generated, has four parts similar and equal, terminated at four cardinal points," as is set forth in detail by Bernouilli.

25. To find the involute of the rectifiable spherical epicycloid.

By § 3 the relations of the coordinates are

$$\cos r' = \cos r \cos R + \sin p \sin R,$$

where $$\tan R = \frac{\sin r\, dr}{\cos p\, dp}, \text{ and } \sin^2 p' = \sin^2 r - \sin^2 p.$$

In this epicycloid $\cos^2 r' + \sin^2 p' = \cos^2 a$.

The differential equation to this involute is

$$\sin^2 a = \cos^2 r + \sin^2 p - (\cos r \cos R + \sin p \sin R)^2.$$

To separate the variables, let $\rho \cos\theta = \cos r$, $\rho \sin\theta = \sin p$

$$\rho^4 (d\theta)^2 = (\rho^2 - \sin^2 a)\{(d\rho)^2 + \rho^2 (d\theta)^2\},$$

$$\rho \frac{d\theta}{d\rho} \sin a = \sqrt{(\rho^2 - \sin^2 a)}.$$

The integral of this equation is the equation to the involute

$$\theta = \operatorname{cosec} a \sqrt{(\rho^2 - \sin^2 a)} - \cos^{-1}\left(\frac{1}{\rho}\sin a\right).$$

After replacing the values of ρ, θ, this becomes

$$\tan^{-1}\left(\frac{\sin p}{\cos r}\right) = \operatorname{cosec} a \sqrt{(\cos^2 r + \sin^2 p - \sin^2 a)} - \cos^{-1}\frac{\sin a}{\sqrt{(\cos^2 r + \sin^2 p)}}.$$

26. To find the equation to the spherical epitrochoid (figs. 1, 2).

Lexell's method determines the elemental arc as in §§ 6, 15,

$$ds = d\phi \sin(a+b) \operatorname{cosec} a \sin PQ',$$

where P is the point of contact of the rolling curve, and Q' a point in the epitrochoid, $\angle QCP = \phi$, perpendicular arc $CN = z$,

$$\cos PQ' = \cos(PN + Q'N)$$

$$= \cos b \cos c \sec^2 z - \sqrt{\{(1 - \cos^2 b \sec^2 z)(1 - \cos^2 c \sec^2 z)\}},*$$

and also

$$= \cos b \cos c + \sin b \sin c \cos \phi.$$

Differentiate both expressions,

$$\frac{d}{d\phi}(PQ') = \sin b \sin c \operatorname{cosec} PQ' \sin \phi = \sin z$$

$$- \sin PQ' \operatorname{cosec} z \frac{d}{dz}(PQ') = 2 \cos b \cos c \sec^3 z$$

$$+ \cos^2 b \sec^3 z \sqrt{\left(\frac{1 - \cos^2 c \sec^2 z}{1 - \cos^2 b \sec^2 z}\right)} + \cos^2 c \sec^3 z \sqrt{\left(\frac{1 - \cos^2 b \sec^2 z}{1 - \cos^2 c \sec^2 z}\right)}.$$

After substituting, the first member of the equation becomes

$$- \frac{ds}{dz} \sin a \operatorname{cosec}(a+b).$$

The relation between p and r may be obtained as in § 15

$$\cos^2 p - \cos^2 r = \sin^2 r \left(\frac{dr}{ds}\right)^2 = \frac{\sin^2 a \sin^2 z}{\sin^2 b} = \frac{\sin^2 a \sin^2 c \sin^2 \phi}{\sin^2 PQ'}$$

$$= \frac{\sin^2 a \{\cos(a+b-c) - \cos r\}\{\cos r - \cos(a+b+c)\}}{\sin^2(a+b) - (\cos c \sin a + \sin b \cos r)^2}.$$

---

* Lexell's expression contains errors in the constants.

27. To find the length of the spherical epitrocloid.

From § 26, $$-\frac{ds}{dz}\sin a\,\mathrm{cosec}\,(a+b)$$

$$=2\cos b\cos c\sec^3 z+\cos^2 b\sec^3 z\sqrt{\left(\frac{1-\cos^2 c\sec^2 z}{1-\cos^2 b\sec^2 z}\right)}$$

$$+\cos^2 c\sec^3 z\sqrt{\left(\frac{1-\cos^2 b\sec^2 z}{1-\cos^2 c\sec^2 z}\right)}.$$

The integral of the first term is

$$\cos b\cos c\{\sec z\tan z+\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}z)\}.$$

The integrals of the last terms may be expressed by elliptic integrals.

Since $$\frac{1-\cos^2 c\sec^2 z}{1-\cos^2 b\sec^2 z}=\frac{\sin^2 c}{\sin^2 b}\cdot\frac{\cot^2 z-\cot^2 c}{\cot^2 z-\cot^2 b},$$

assume $$\cot^2 z=\cot^2 b+\sec^2\phi(\cot^2 c-\cot^2 b)=\cot^2 b+m\sec^2\phi$$

$$-\cot z\,\mathrm{cosec}^2 z\,dz=m\sec^2\phi\tan\phi\,d\phi.$$

The integral of the second term is

$$-m\cos^2 b\sin c\,\mathrm{cosec}\,b\int d\phi\tan^2\phi\sec\phi\tan^4 z\,\mathrm{cosec}\,z$$

$$=-m\cos b\cot b\sin c\int d\phi\frac{\sin^2\phi}{(m+\cot^2 b\cos^2\phi)^2}(m+\mathrm{cosec}^2 b\cos^2\phi)^{\frac{1}{2}}.$$

This integral may be made to depend by formulæ of reduction upon elliptic functions of the three orders (Hymers' *Integral Calculus*, Art. 153).

The integral of the third term is precisely similar.

Cor. If c or b be 90°, as in the case of the rectifiable epicycloid,

$$-\tan a\frac{ds}{dz}=\frac{\cos^2 c}{\cos^2 z}\sqrt{\frac{1}{(\sin^2 c-\sin^2 z)}},$$

whose integral is an elliptic function of the third order.

28. On the quadrature of the spherical epitrochoid.*

The limits here selected are analogous to those which have been taken for the epicycloid, viz. the epitrochoid itself, a circle concentric with the fixed circle through the lowest apse of the roulette, and (1) the circle concentric with the

* The limits chosen by Lexell are the epitrochoid, the fixed circle, and the arc PQ' (the radius of instantaneous motion), in a given position and the same arc in its initial position.

generating circle through the moving point, and (2) the vector arc drawn from the fixed centre to the moving point.

In the first case, the area as in § 7 is

$$\int\{\cos(a+b-c)-\cos r\}\,d\theta = \operatorname{cosec} a \sin b \times \int d\phi\,\{\cos(a+b-c)-\cos(a+b)\cos c-\sin(a+b)\sin c\cos\phi\},$$
$$= \operatorname{cosec} a \sin b \sin c \sin(a+b)(\phi-\sin\phi),$$

where, as usual, $\theta = AOP$, $\phi = PCQ'$, $\chi = AOQ'$.

The area in the second case is

$$\int_{a+b-c}^{r}\int_{0}^{\chi} \sin r\,dr\,d\chi = \int_{0}^{\chi}\{\cos(a+b-c)-\cos r\}\,d\chi.$$

As in § 17,

$$\cos r = \cos(a+b)\cos c+\sin(a+b)\sin c\cos\phi$$
$$= \cos(a+b-c)-\sin(a+b)\sin c(1-\cos\phi),$$
$$\cos(\theta-\chi)\sin(a+b) = \cos c\operatorname{cosec} r-\cos(a+b)\cot r,$$
$$\sin(\theta-\chi)\sin r = \sin\phi\sin c,\quad \frac{dr}{d\phi} = \sin(a+b)\sin(\theta-\chi),$$
$$\frac{d\chi}{d\phi} = \sin b\operatorname{cosec} a-\{\cos c\cos r-\cos(a+b)\}\operatorname{cosec}^2 r.$$

The area required is

$$\int_0^{\phi} d\phi\,[\{\cos(a+b-c)-\cos r\} \times\{\sin b\operatorname{cosec} a-[\cos c\cos r-\cos(a+b)]\operatorname{cosec}^2 r\}]$$
$$= \sin b\operatorname{cosec} a\int_0^{\phi} d\phi\,\{\cos(a+b-c)-\cos r\}-\int_0^{\phi} d\phi\cos c$$
$$+\tfrac{1}{2}\{\cos c-\cos(a+b)\}\{1-\cos(a+b-c)\}\int_0^{\phi}\frac{d\phi}{1-\cos r}$$
$$+\tfrac{1}{2}\{\cos c+\cos(a+b)\}\{1+\cos(a+b-c)\}\int_0^{\phi}\frac{d\phi}{1+\cos r}$$
$$= \operatorname{cosec} a\sin b\sin c\sin(a+b)(\phi-\sin\phi)-\phi\cos c$$
$$+[1-\cos(a+b-c)] \times\tan^{-1}\{\sin\tfrac{1}{2}(a+b+c)\operatorname{cosec}\tfrac{1}{2}(a+b-c)\tan\tfrac{1}{2}\phi\}$$
$$+[1+\cos(a+b-c)] \times\tan^{-1}\{\cos\tfrac{1}{2}(a+b+c)\sec\tfrac{1}{2}(a+b-c)\tan\tfrac{1}{2}\phi\}$$
$$= \phi\{\operatorname{cosec} a\sin b\sin c\sin(a+b)-\cos c\}$$
$$-\sin\phi\sin b\sin c(\cos b+\cot a\sin b)$$
$$+\sin^{-1}\left\{\frac{\sin(a+b)\sin\phi}{\sin r}\right\}-\cos(a+b-c)\sin^{-1}\left(\frac{\sin c\sin\phi}{\sin r}\right).$$

And $$\sin r \sin OQ'C = \sin(a+b)\sin\phi,$$
$$\sin r \sin COQ' = \sin c \sin\phi,$$

as in § 17.

For a half-roll, $\phi = \pi$, and half the area of the epitrochoid from apse to apse is

$$\pi\{1 + \sin b \sin c (\sin b \cot a + \cos b) - \cos c\}.$$

The difference of the areas in the two cases is $\pi(1 - \cos c)$, which is half the spherical surface intercepted by the moving circle.

29. The radius of curvature of the spherical epitrochoid may be found as in § 19 in terms of the coordinates p and r from the relations

$$\cos r = \cos a \cos PQ' - \sin a \sin PQ' \cos CPQ',$$
$$\sin p = \cos a \sin PQ' + \sin a \cos PQ' \cos CPQ',$$

$$\cot R = \frac{\cos p}{\sin r}\frac{dp}{dr}, \quad \cos CPQ' \sin b \sin PQ' = \cos c - \cos b \cos PQ'.$$

The evolute of the epitrochoid may be found as in § 20.

30. Bernouilli remarks that the apparent path of the sun's centre is an epitrochoid, affiliated to the rectifiable epicycloid.

"If the sun were to move in the ecliptic with a velocity equal to that which the Tropic of Capricorn has to execute the diurnal motion, and so the time of a revolution of the sun in the ecliptic were to the time of a revolution of the sphere, which constitutes the length of the natural day, in the same ratio, which the radius of ecliptic (or of the sphere) bears to the radius of the tropic as $1 : \cos 23^\circ 30'$, the centre of the sun would exactly describe a rectifiable epicycloid. But as the sun has its proper motion in the ecliptic much slower, the line, which the centre of the sun describes between the two tropics during the space of a year by the combined common and proper motion will be a *Cycloide allongée*, rather than a spiral, under the form that Tycho conceived."

31. These theorems are applicable to hypocycloids and hypotrochoids, if (b) the radius of the rolling circle be changed in sign.

32. The preceding paragraphs constitute a first chapter on spherical roulettes. It would be interesting and not difficult to extend the subject, and to write also on Spherical Glissettes, by following the guidance of Mr. Besant's valuable monograph in Plane Geometry.

Falmouth,
Aug. 21, 1882.

ON THE FORCES EXPERIENCED BY A SOLID MOVING IN AN INFINITE MASS OF LIQUID.

By H. Lamb.

IN the methods by which Thomson and Kirchhoff have severally investigated the equations of motion of a solid in an infinite mass of liquid, the fluid and the solid are treated as forming together one dynamical system, and in this way the calculation of the effect of the fluid pressure on the surface of the solid is evaded. Indeed, the only piece of hydrodynamical knowledge which is required in these methods is the fact that the motion of the fluid (supposed originally at rest) is irrotational, and therefore fully determinate when that of the solid is known. By comparing the results obtained in this way with the equations of motion of the solid alone, we are led to certain expressions for the components of the force ($\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$) and couple ($\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$) due to the pressures exerted on the solid by the surrounding fluid; viz. we have*

$$\mathfrak{X} = -\frac{d}{dt}\frac{d\mathfrak{T}}{du} + r\frac{d\mathfrak{T}}{dv} - q\frac{d\mathfrak{T}}{dw}, \text{ \&c., \&c. ..... (1),}$$

$$\mathfrak{L} = -\frac{d}{dt}\frac{d\mathfrak{T}}{du} + w\frac{d\mathfrak{T}}{dv} - v\frac{d\mathfrak{T}}{dw} + r\frac{d\mathfrak{T}}{dq} - q\frac{d\mathfrak{T}}{dr}, \text{ \&c., \&c. ...(2).}$$

The symbols u, v, w and p, q, r here express the linear and angular motion of the solid relatively to a system of rectangular axes fixed in the solid and moving with it, and $\mathfrak{T}$ denotes the kinetic energy of the *fluid* expressed as a quadratic function of these six quantities.

It may be worth while to shew, by way of verification, how the expressions (1) and (2) may be obtained by direct calculation from the ordinary hydrodynamical formulæ

$$\frac{p}{\rho} = \text{const.} - \left[\frac{d\Psi}{dt}\right] - \tfrac{1}{2}Q^2 \text{ ..... (3),}$$

---

* *Motion of Fluids*, § 119.

where p, ρ, ϕ have their usual meanings, and

$$Q^2 = \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2.$$

It will be convenient to begin by stating a few analytical results by which the final reduction is greatly simplified. The velocity-potential ϕ is of the form

$$\phi = u\phi_1 + v\phi_2 + w\phi_3 + p\chi_1 + q\chi_2 + r\chi_3,$$

where the functions ϕ_1, χ_1, &c., are determined by the conditions that they satisfy Laplace's equation throughout the space occupied by the fluid, that their derivatives vanish at infinity, and that at the surface of the solid

$$\frac{d\phi}{d\nu} = l(u + qz - ry) + m(v + rx - pz) + n(w + py - qx) \ldots (4),$$

and thence

$$\frac{d\phi_1}{d\nu_1} = l, \text{ &c., &c.,}$$

$$\frac{d\chi_1}{d\nu} = ny - mz, \text{ &c., &c.}$$

In these formulæ $d\nu$ denotes an element of the normal, drawn towards the fluid, at any point of the surface of the solid, and l, m, n are its direction-cosines. Now

$$2\mathfrak{T} = -\rho \iint \phi \frac{d\phi}{d\nu} dS,$$

the integration extending over the surface of the solid; and hence

$$\frac{d\mathfrak{T}}{du} = -\rho \iint \phi \frac{d\phi_1}{d\nu} dS = -\rho \iint l\phi dS \ldots\ldots\ldots\ldots\ldots (5),$$

$$\frac{d\mathfrak{T}}{dp} = -\rho \iint \phi \frac{d\chi_1}{d\nu} dS = -\rho \iint \phi (ny - mz)\, dS \ldots\ldots\ldots (6).$$

Again, the symbol $\left[\frac{d\phi}{dt}\right]$ in (3) denotes the rate at which ϕ is changing at a point *fixed in space*. If we use $\frac{d\phi}{dt}$ to denote the rate of change of ϕ for a point *moving with the solid*, viz.

$$\frac{d\phi}{dt} = \phi_1 \frac{du}{dt} + \phi_2 \frac{dv}{dt} + \phi_3 \frac{dw}{dt} + \chi_1 \frac{dp}{dt} + \chi_2 \frac{dq}{dt} + \chi_3 \frac{dr}{dt},$$

we have

$$\frac{d\phi}{dt} = \left[\frac{d\phi}{dt}\right] + (u + qz - ry)\frac{d\phi}{dx} + (v + rx - pz)\frac{d\phi}{dy} + (w + py - qx)\frac{d\phi}{dz} \quad \ldots\ldots\ldots(7).$$

Lastly, we shall also require the following formulæ, which are readily proved after the manner of Green's Theorem:

$$\tfrac{1}{2}\iint Q^2 l\, dS = \iint \frac{d\phi}{dx}\,\frac{d\phi}{d\nu}\, ds \quad \ldots\ldots\ldots(8),$$

$$\tfrac{1}{2}\iint Q^2 (ny - mz) = \iint \left(y\frac{d\phi}{dz} - z\frac{d\phi}{dy}\right)\frac{d\phi}{d\nu}\, dS \quad \ldots\ldots(9),$$

$$\iint \left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = 0 \quad \ldots\ldots\ldots\ldots\ldots(10),$$

$$\iint x\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = 0 \quad \ldots\ldots\ldots\ldots\ldots(11),$$

$$\iint y\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = -\iint \phi n\, dS \quad \ldots\ldots\ldots(12),$$

$$\iint z\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = \iint \phi m\, dS \quad \ldots\ldots\ldots(13),$$

$$\iint x^2\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = 0 \quad \ldots\ldots\ldots\ldots\ldots(14),$$

$$\iint xy\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = -\iint \phi\, nx\, dS \quad \ldots\ldots\ldots(15),$$

$$\iint xz\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS = \iint \phi\, mx\, dS \quad \ldots\ldots\ldots(16).$$

The number attached to any one of these formulæ will be taken to represent as well the two others which can be written down from it by symmetry.

Proceeding now to the calculation of $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, &c., we have

$$\mathfrak{X} = -\iint p l dS.$$

Substituting the value of p from (3) eliminating $\left[\frac{d\phi}{dt}\right]$ by means of (7), and bearing in mind the relations (8) and (4), we find

$$\begin{aligned}\mathfrak{X} &= \rho \iint \frac{d\phi}{dt} l dS \\ &\quad + \iint \left\{ (v + rx - pz)\left(m\frac{d\phi}{dx} - l\frac{d\phi}{dy}\right) \right. \\ &\quad \left. + (w + py - qx)\left(n\frac{d\phi}{dx} - l\frac{d\phi}{dz}\right) \right\} dS \\ &= -\frac{d}{dt}\frac{d\mathfrak{T}}{du} + r\frac{d\mathfrak{T}}{dv} - q\frac{d\mathfrak{T}}{dw},\end{aligned}$$

by (5), (10), (11). This agrees with (1).

Again

$$\begin{aligned}\mathfrak{L} &= -p \iint (ny - mz)\, dS \\ &= \rho \iint \frac{d\phi}{dt}(ny - mz)\, dS \\ &\quad + \rho \iint \left(y\frac{d\phi}{dz} - z\frac{d\phi}{dy}\right) \frac{d\phi}{dv} dS \\ &\quad - \rho \iint (ny - mz) \left\{ (u + qz - ry)\frac{d\phi}{dx} \right. \\ &\quad \left. + (v + rx - pz)\frac{d\phi}{dy} + (w + py - qz)\frac{d\phi}{dz} \right\} dS,\end{aligned}$$

by (7) and (9). The first of these integrals $= -\frac{d}{dt}\frac{d\mathfrak{T}}{dp}$; and if we

substitute the value of $\frac{d\phi}{dv}$ from (4), the last two integrals will reduce to

$$\rho \iint (u + qz - ry) \left\{ y \left(l \frac{d\phi}{dz} - n \frac{d\phi}{dr} \right) - z \left(l \frac{d\phi}{dy} - m \frac{d\phi}{dx} \right) \right\} dS$$

$$+ \rho \iint (v + rx - pz) \; y \left(m \frac{d\phi}{dz} - n \frac{d\phi}{dy} \right) dS$$

$$+ \rho \iint (w + py - qx) \; z \left(m \frac{d\phi}{dz} - n \frac{d\phi}{dy} \right) dS.$$

In this expression the coefficient of u vanishes by (13). The coefficient of $v = \iint \phi n dS$, by (11), *i.e.* it $= -\frac{d\mathfrak{T}}{dw}$. Similarly the coefficient of w is $\frac{d\mathfrak{T}}{dv}$. The coefficient of p vanishes identically. The coefficient of q

$$= \rho \iint yz \left(l \frac{d\phi}{dz} - n \frac{d\phi}{dx} \right) dS$$

$$- \rho \iint zx \left(m \frac{d\phi}{dz} - n \frac{d\phi}{dy} \right) dS$$

$$- \rho \iint z^2 \left(l \frac{d\phi}{dy} - m \frac{d\phi}{dx} \right) dS.$$

The last of these integrals is zero, by (14), and the first two are, by (15) and (16),

$$= \rho \iint \phi (mx - ly) \, dS$$

$$= -\frac{d\mathfrak{T}}{dr}.$$

Similarly, the coefficient of r is $\frac{d\mathfrak{T}}{dq}$. Collecting our results, we obtain the formula (2).

The adaptation of the foregoing processes to the case where there is cyclic irrotational motion through one or more apertures in the solid presents no difficulty. The results will be found to agree with those deduced from Thomson's theory of the "Impulse."

EXPRESSIONS FOR $\arg \operatorname{sn} a$ AND $(\arg \operatorname{sn} a)^2$ AS DEFINITE INTEGRALS.

By J. W. L. Glaisher.

§ 1. THE two following equations, in which a is supposed to be less than unity, give expressions for $\arg \operatorname{sn} a$ and $(\arg \operatorname{sn} a)^2$ as definite integrals.

$$\arg \operatorname{sn} a = \frac{1}{\pi} \int_0^{K'} \log \left\{ \frac{1 + a \operatorname{dn}(u, k')}{1 - a \operatorname{dn}(u, k')} \right\} du \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(\arg \operatorname{sn} a)^2 = \frac{2}{\pi} \left[K' \int_0^K \log \{1 - a^2 k^2 \operatorname{sn}^2 u\} \, du - K \int_0^{K'} \log \{1 - a^2 \operatorname{dn}^2 (u, k')\} \, du \right] \ldots\ldots (2).$$

Since

$$\int_0^K \operatorname{dn} u \, du = \int_0^K \operatorname{dn} (K - u) \, du = \int_0^K \frac{k'}{\operatorname{dn} u} \, du,$$

the first formula may be also written

$$\arg \operatorname{sn} a = \frac{1}{\pi} \int_0^{K'} \log \left\{ \frac{\operatorname{dn}(u, k') + ak}{\operatorname{dn}(u, k') - ak} \right\} du.$$

§ 2. The formulæ (1) and (2) were obtained by showing that the coefficients of a^{2n+1} and a^{2n} in the expansions of $\arg \operatorname{sn} a$ and $(\arg \operatorname{sn} a)^2$ were respectively equal to

$$\frac{2}{\pi} \cdot \frac{1}{2n+1} \int_0^{K'} \operatorname{dn}^{2n+1} (u, k') \, du$$

and

$$\frac{2}{\pi} \cdot \frac{1}{2n} \left(K \int_0^{K'} \{\operatorname{dn}(u, k')\}^{2n} \, du - K' \int_0^K \{k \operatorname{sn} u\}^{2n} \, du \right).$$

The formula (1) may however be readily established as follows: it is easy to see that

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\Delta^2 \alpha - k'^2 \sin^2 \alpha \sin^2 \phi} = \frac{\pi}{2 \cos \alpha \, \Delta \alpha},$$

whence

$$\frac{1}{\pi} \int_0^{\frac{1}{2}\pi} \frac{2 \cos \alpha}{\Delta^2 \alpha - k'^2 \sin^2 \alpha \sin^2 \phi} \, d\phi = \frac{1}{\Delta \alpha}.$$

Integrating with respect to α, we find

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log\left\{\frac{1+\sin\alpha\sqrt{(\sin^2\phi+k^2\cos^2\phi)}}{1-\sin\alpha\sqrt{(\sin^2\phi+k^2\cos^2\phi)}}\right\}\frac{d\phi}{\sqrt{(\sin^2\phi+k^2\cos^2\phi)}}$$
$$=\int_0^{\alpha}\frac{d\alpha}{\Delta\alpha}=F(\alpha),$$

which on putting $\sin\alpha=a$ and $\phi=\operatorname{am}(u,k')$ gives (1).

§ 3. The formulæ (1) and (2) are true when $\alpha=1$, for in this case they give

$$\pi K = \int_0^{K'}\log\{1+\operatorname{dn}(u,k')\}\,du-\int_0^{K'}\log\{1-\operatorname{dn}(u,k')\}\,du,$$

$$\tfrac{1}{4}\pi K^2=K'\int_0^{K}\log\operatorname{dn}u\,du-K\int_0^{K'}\log\{k'\operatorname{sn}(u,k')\}\,du;$$

which are readily verified by means of the formulæ*

$$\int_0^K\log(1+\operatorname{dn}u)\,du=\tfrac{1}{4}\pi K'+\tfrac{1}{2}K\log k,$$

$$\int_0^K\log(1-\operatorname{dn}u)\,du=-\tfrac{3}{4}\pi K'+\tfrac{1}{2}K\log k,$$

$$\int_0^K\log\operatorname{sn}u\,du=-\tfrac{1}{4}\pi K'-\tfrac{1}{2}K\log k,$$

$$\int_0^K\log\operatorname{dn}u\,du=\tfrac{1}{2}K\log k'.$$

§ 4. The formula (2) may be transformed so as to assume the form

$$(\operatorname{arg\,sn}a)^2=\frac{2i}{\pi}\left[K\int_0^{K+iK'}\log(1-a^2k^2\operatorname{sn}^2u)\,du\right.$$
$$\left.-(K+iK')\int_0^K\log(1-a^2k^2\operatorname{sn}^2u)\,du\right].$$

In this expression we may regard $\log(1-a^2k^2\operatorname{sn}^2u)$ as equivalent to the series

$$-a^2k^2\operatorname{sn}^2u-\tfrac{1}{2}a^4k^4\operatorname{sn}^4u-\tfrac{1}{3}a^6k^6\operatorname{sn}^6u-\&c,$$

the coefficient of u^{2n} in the expansion of $(\operatorname{arg\,sn}u)^2$ being

$$\frac{2i}{\pi}\frac{k^{2n}}{2n}\left[(K+iK')\int_0^K\operatorname{sn}^{2n}u\,du-K\int_0^{K+iK'}\operatorname{sn}^{2n}u\,du\right].$$

* *Proceedings of the Royal Society*, vol. XXIX. pp. 333, 336 (1879).

§ 5. The following forms involving double integration may also be noticed:

$$(\arg \operatorname{sn} a)^2 = \frac{2}{\pi} \int_0^K \int_0^{K'} \log \left\{ \frac{1 - a^2 k^2 \operatorname{sn}^2 u}{1 - a^2 \operatorname{dn}^2 (v, k')} \right\} du\, dv,$$

$$= \frac{2i}{\pi} \int_0^{K+iK'} \int_0^K \log \left\{ \frac{1 - a^2 k^2 \operatorname{sn}^2 u}{1 - a^2 k^2 \operatorname{sn}^2 v} \right\} du\, dv,$$

$$= \frac{2}{\pi} \int_0^1 \int_0^1 \log \left\{ \frac{1 - a^2 k^2 x^2}{1 - a^2 + a^2 k'^2 y^2} \right\} \frac{dx\, dy}{\sqrt{(1 - x^2 . 1 - y^2 . 1 - k^2 x^2 . 1 - k'^2 y^2)}},$$

$$= \frac{2i}{\pi} \int_0^{\frac{1}{k}} \int_0^1 \log \left\{ \frac{1 - a^2 k^2 x^2}{1 - a^2 k^2 y^2} \right\} \frac{dx\, dy}{\sqrt{(1 - x^2 . 1 - y^2 . 1 - k^2 x^2 . 1 - k^2 y^2)}}.$$

§ 6. By transforming the integral in (1) we may obtain the formula

$$\arg \operatorname{sn} a = \frac{i}{\pi} \int_K^{K+iK'} \log \left\{ \frac{1 - ak \operatorname{sn} u}{1 + ak \operatorname{sn} u} \right\} du \ldots\ldots\ldots (3),$$

in which however the track of the integration is not arbitrary.

§ 7. The following formula for $\arg \operatorname{cn} a$ and $\arg \operatorname{dn} a$ may also be noticed:

$$\arg \operatorname{cn} a = K - \frac{2}{\pi} \int_0^K \operatorname{arc\,tan} \left(\frac{ak \operatorname{cn} u}{k'} \right) du$$

$$- \frac{1}{\pi} \int_0^{K'} \log \left\{ \frac{1 + a \operatorname{cn} (u, k')}{1 - a \operatorname{cn} (u, k')} \right\} du,$$

$$= K - \frac{2}{\pi} \int_0^{K+iK'} \operatorname{arc\,tan} \left(\frac{ak \operatorname{cn} u}{k'} \right) du.$$

$$\arg \operatorname{dn} a = K + iK' - \frac{i}{\pi} \int_0^K \log \left\{ \frac{\operatorname{dn} u + a}{\operatorname{dn} u - a} \right\} du,$$

$$= K + iK' + \frac{1}{\pi} \int_{K'}^{K'+iK} \log \left\{ \frac{1 - a \operatorname{sn} (u, k')}{1 + a \operatorname{sn} (u, k')} \right\} du.$$

In the expressions for $\arg \operatorname{cn} a$, a is not to exceed the smaller of the quantities 1 and $\frac{k'}{k}$; and in the expressions for $\arg \operatorname{dn} a$, a is not to exceed k'.

The two expressions for $\arg \operatorname{sn} a$ are readily converted the one into the other by means of the formula

$$\int_0^{K+iK'} \{k \operatorname{cn} u\}^{2n+1}\, du = \int_0^{K} \{k \operatorname{cn} u\}^{2n+1}\, du + (-)^n \int_0^{K'} \{k' \operatorname{cn}(u, k')\}^{2n+1}\, du.$$

ON SYSTEMS OF CIRCLES AND BICIRCULAR QUARTICS.

By Homersham Cox, B.A.

IT is not very long since it has been recognised that the validity of algebraical results depends solely on the formal laws of combination of the symbols employed, and not on any geometrical or arithmetical interpretation of those symbols. As algebra arose from arithmetic, the early mathematicians confused the arithmetic and algebraic point of view. For example, negative quantities were rejected till Descartes showed how to interpret them, and even in this century imaginary quantities were used with hesitation. It is now, however, acknowledged that an algebraical result is formally true so long as the laws of combination are observed. Whether it has any interpretation applicable to the problem in hand is another and independent question. When this principle was adopted, it was natural to enquire whether other relations besides numerical relations could be expressed in a simple symbolic form. The first steps in this direction were made by Sir W. Hamilton, and Grassmann about the same time. Soon after, in his *Logical Calculus*, Boole gave another example of mathematical laws applied to symbols with no numerical meaning.

This paper is intended to apply the same principles to the treatment of circles. As far as regards addition, the method is indeed implied in the abridged notation of ordinary analytical geometry and is established on an independent basis by Grassmann. There are also papers following out the ideas of Grassmann by Mehmke in the "Zeitschrift für Mathematik und Physik," and Prof. Clifford in the volume containing his reprinted papers. By none of these writers, however, is the conception of a product of circles introduced.

In the first section I have defined the addition of circles and shown that this operation obeys the associative and commutative laws. This section might have been shortened by introducing Cartesian methods, but it seemed better to establish the whole independently.

In the second section the product of two circles is defined, and the operation is shown to satisfy the commutative and distributive laws. The associative law does not come in question, since the product of more than two circles is never considered. The special cases when the circles reduce to lines or points are examined. The addition of lines is shown to be included in the addition of circles.

Some well-known properties of circles are thus proved as an illustration of the method. They are taken for the most part from Dr. Casey's "Sequel to Euclid." It is shown that three circles passing through a point have properties analogous to those of an ordinary triangle. A nine-point circle is found for the three circles, and it is shown to touch the inscribed and escribed circles. The circles touching three given circles are examined. Dr. Hart's extension of Feuerbach's theorem is proved. The proof is indeed not so short as Dr. Casey's geometrical one, but then no analytical method can always equal geometrical methods in simplicity. It is shown that Dr. Hart's circle cuts the original circles at angles equal to the differences of the angles they make with one another. The theorem that the circles circumscribing the triangles obtained from a quadrilateral by leaving out each side in turn pass through a point is extended to the case of circles passing through a point. An easy proof of Miquel's theorem is thus obtained and an indefinite series of other theorems. The next section contains the reduction of the general equation of a bicircular quartic, and the last the proof of some of the principal theorems given in Salmon's *Higher Plane Curves* relating to these quartics.

The Addition of Circles.

It is shown in the "Ausdehnungslehre" that the sum of multiples of two points is a multiple of a point on the line joining them, and that the sum of two straight lines is a line passing through their point of intersection. By analogy then the sum of multiples of two intersecting circles is some multiple of a circle passing through their two points of intersection. We complete this determination of the circle

by assuming that if C_1, C_2, C_3 be three circles and O_1, O_2, O_3 their centres, then when

$$pC_1 + qC_2 = (p+q)\,C_3,$$
$$pO_1 + qO_2 = (p+q)\,O_3,$$

where the addition of points has the same meaning as in the "Ausdehnungslehre" or the Barycentric Calculus.

It follows that

$$(pC_1 + qC_2) + (rC_1 + sC_2) = (p+r)\,C_1 + (q+s)\,C_2,$$

for we should get the same centre for both circles, and all the circles pass through the same two points.

We will determine the radius r_3 of C_3 in terms of the radii r_1, r_2 of C_1, C_2. Let A (fig. 4) be one of the points through which all the circles pass, then

$$r_1^2 = O_1A^2 = O_1O_3^2 + r_3^2 - 2rO_1O_3 \cos O_1O_3A,$$
$$r_2^2 = O_2A^2 = O_2O_3^2 + r_3^2 + 2rO_2O_3 \cos O_1O_3A.$$

But, from the principles of Barycentric Calculus,

$$pO_1O_3 = qO_3O_2 = \frac{pq}{p+q}\,d,$$

if $d = O_1O_2$, the distance between the centres of the two circles.

Therefore

$$pr_1^2 + qr_2^2 = \frac{pq}{(p+q)}\,d^2 + (p+q)\,r_3^2,$$

or

$$(p+q)^2 r_3^2 = p^2 r_1^2 + q^2 r_2^2 + pq\,(r_1^2 + r_2^2 - d^2).$$

We may adopt this analytical formula even when the circles do not intersect, and then we shall always have

$$(pC_1 + qC_2) + (rC_1 + OC_2) = (p+r)\,C_1 + (q+s)\,C_2;$$

for since this is true for intersecting circles, the expression for the radii of the two circles must be identical, and therefore it is true for non-intersecting circles.

When

$$r_1 = 0,\ r_2 = 0,\ (p+q)^2 r_3^2 = -pqd^2 \text{ or } r_3^2 = O_3O_1 \cdot O_3O_2.$$

In this case C_1C_2 reduce to points, and it is seen that the sum of two points is not a point but a circle of real or imaginary radius, as p, q are of opposite or the same sign.

If $r_3 = 0$,

$$p^2 r_1^2 + q^2 r_2^2 + pq\,(r_1^2 + r_2^2 - d^2) = 0.$$

The roots of this quadratic equation can only be real when

$$(r_1^2 + r_2^2 - d^2)^2 > 4 r_1^2 r_2^2,$$

that is $$r_1^2 + r_2^2 - d^2 > 2 r_1 r_2,$$

or $$r_1^2 + r_2^2 - d^2 < -2 r_1 r_2,$$

$$d^2 < (r_1 - r_2)^2 \text{ or } > (r_1 + r_2)^2.$$

That is to say, if real points are to belong to the system derived from two circles, one circle must lie either entirely inside or entirely outside the other. When the circles touch there is only one real point belonging to the system, and it is seen to be the point of contact. Another special case is when $p + q = 0$, then r_3 becomes infinite, but $(p+q)\,r_3$ has the limiting value pd. The centre O_3 passes off to infinity. $C_2 - C_1$ may therefore be considered to represent a straight line of length d perpendicular to $O_1 O_2$.

If L be the point where this line cuts O_1, O_2, then

$$O_1 L = \text{limit of } (O_1 O_3 - r_3)$$

$$= \text{limit of } \frac{1}{p+q}\left[qd - \sqrt{\{q^2 d^2 + (p+q)\,(pr_1^2 + qr_2^2 - qd^2)\}}\right]$$

$$= \text{limit of } \frac{qd^2 - pr_1^2 - qr_2^2}{2qd}$$

$$= \frac{d^2 + r_1^2 - r_2^2}{2d}$$

$$O_2 L = \frac{d^2 + r_2^2 - r_1^2}{2d},$$

and hence

$$O_1 L^2 - O_2 L^2 = r_1^2 - r_2^2,$$

$$O_1 L^2 - r_1^2 \quad = O_2 L^2 - r_2^2.$$

If T be any point on the line $C_2 - C_1$, and TP, TQ tangents to the circles C_1, C_2, then

$$TP^2 = O_1 L^2 + TL^2 - r_1^2 = TQ^2,$$

or the line $C_2 - C_1$ is the locus of points from which equal tangents to the two circles can be drawn. If the circles intersect it will pass through the points of intersection.

Let there now be three circles C_1, C_2, C_3, with centres A, B, C, so that neither of the circles can be derived from the other two (fig. 5).

Let $$(\beta+\gamma)\,C' = \beta C_2 + \gamma C_3,$$

and let D be the centre of this circle, r' its radius; r_1, r_2, r_3 the radii of C_1, C_2, C_3; a, b, c the sides of the triangle ABC. Then

$$(\beta+\gamma)^2 r'^2 = \beta^2 r_2^2 + \gamma^2 r_3^2 + \beta\gamma\,(r_2^2 + r_3^2 - a^2).$$

If $(\alpha+\beta+\gamma)\,C = \alpha C_1 + (\beta+\gamma)\,C' = \alpha C_1 + \beta C_2 + \gamma C_3$,

and P be the centre r, the radius of C,

$$(\alpha+\beta+\gamma)^2 r^2 = \alpha^2 r_1^2 + (\beta+\gamma)^2 r'^2 + \alpha\,(\beta+\gamma)\,(r_1^2 + r'^2 - AD^2).$$

Now $$AD^2 + BD^2 + 2PD \,.\, OB \cos ADB = c^2,$$
$$AD^2 + CD^2 - 2OD \,.\, CD \cos ADB = b^2,$$

and $$\beta BD = \gamma DC = \frac{\beta\gamma}{\beta+\gamma}\,a,$$

hence $$(\beta+\gamma)\,AD^2 = \beta c^2 + \gamma b^2 - \frac{\beta\gamma}{\beta+\gamma}\,a^2,$$

$$(\beta+\gamma)\,(r'^2 - AD^2) = \frac{\beta^2 r_2^2 + \gamma^2 r_3^2 + \beta\gamma\,(r_2^2 + r_3^3)}{\beta+\gamma} - \beta c^2 - \gamma b^2$$
$$= \beta r_2^2 + \gamma r_3^2 - \beta c^2 + \gamma b^2.$$

It follows that

$$(\alpha+\beta+\gamma)^2 r^2 = \alpha^2 r_1^2 + \beta^2 r_2^2 + \gamma^2 r_3^2 + \beta\gamma\,(r_2^2 + r_3^2 - a^2)$$
$$+ \gamma\alpha\,(r_3^2 + r_1^2 - b^2) + \alpha\beta\,(r_1^2 + r_2^2 - c^2).$$

This expression for r is symmetrical, and therefore the same result would be obtained by adding the circle

$$(\alpha+\beta)\,C'' = \alpha C_1 + \beta C_2 \text{ to } \gamma C_3.$$

The centre P also would be in both cases the same, so that the addition of circles is an associative operation.

From three circles C_1, C_2, C_3, whose centres do not lie in a straight line, a circle may be derived with any point for centre by choosing proper values for the ratios of α, β, γ; but when the centre is given, the radius will be determined by the equation written above. From four circles C_1, C_2, C_3, C_4 any circle $\alpha C_1 + \beta C_2 + \gamma C_3 + \delta C_4$ may in general be derived, since the new ratio $\frac{\delta}{\alpha}$ will be sufficient to give the radius any required value.

Product of Circles.

If to the equation

$$(\beta+\gamma)\,(r'^2 - AD^2) = \beta r_2^2 + \gamma r_3^2 - \beta c^2 - \gamma b^2,$$

we add $(\beta+\gamma)\,r_1^2$, then

$$(\beta+\gamma)(r_1^2 + r'^2 - AD^2) = \beta\,(r_1^2 + r_2^2 - c^2) + \gamma\,(r_1^2 + r_3^2 - b^2).$$

We may then consider the difference between the sum of the squares of the radii of two circles, and the square of the distance between the centres (or any multiple of that difference) as the product of the circles.

For putting (since $\frac{1}{2}$ is the most convenient multiple)

$$C_1C_2 = \tfrac{1}{2}\,(r_1^2 + r_2^2 - c^2), \quad C_1C_3 = \tfrac{1}{2}\,(r_1^2 + r_3^2 - b^2)$$
$$CC' = \tfrac{1}{2}(r^2 + r'^2 - AD^2$$

$$C_1.\beta C_2 = \beta C_1C_2, \quad C.\gamma C_3 = \gamma C_1C_3, \quad C_1(\beta+\gamma)\,C' = (\beta+\gamma)\,CC',$$

the above equation may be written

$$C_1(\beta+\gamma)\,C' = \beta C_1C_2 + \gamma C_1C_3,$$

or
$$C_1(\beta C_2 + \gamma C_3) = C_1\beta C_2 + C_1\gamma C_3.$$

Thus the distributive law holds, and this justifies our regarding $C_1C_2 = \frac{1}{2}\,(r_1^2 + r_2^2 - c^2)$ as a product of two circles.

If the circles cut, and θ be the angle between them,

$$C_1C_2 = r_1r_2\cos\theta.$$

This is analogous to the scalar product of two lines, for that product is equal to the product of their lengths into the cosine of the angle between them. When the circles coincide, $C=0$, $r_1 = r_2$; therefore

$$C_1^2 = r_1^2,$$

or, the square of a circle is equal to the square of its radius. If

$$(\alpha+\beta+\gamma+\delta+\varepsilon+\ldots)\,C = \alpha C_1 + \beta C_2 + \gamma C_3 + \delta C_4 + \varepsilon C_5 + \ldots,$$

and r be the radius of C, r_1, r_2, $r_3\ldots$ of C_1, C_2, $C_3\ldots$, a_{12}, $a_{23}\ldots$ the distances between their centres, then squaring

$$(\alpha+\beta+\gamma+\delta+\varepsilon+\ldots)\,r^2 = \alpha^2r_1^2 + \beta^2r_2^2 + \ldots$$
$$+ \alpha\beta\,(r_1^2 + r_2^2 - a_{12}^2) + \alpha\gamma\,(r_1^2 + r_3^2 - a_{13}^2) + \ldots\,.$$

This formula gives the radius of a circle derived from any number of given circles and includes those previously found as particular cases.

It was seen that tangents drawn from any point of the line $C_1 - C_2$ to the circles C_1 and C_2 are equal. Similarly, tangents drawn from any point of $C_1 - C_3$ to the circles C_1 and C_3 are equal. Therefore the tangents drawn from the point where these lines intersect to all three circles are equal; and a circle can be drawn with this point for centre and the tangent for radius cutting the three circles at right angles.

The radius r of this circle is given by the relation connecting the distances $r^2 + r_1^2$, $r^2 + r_2^2$, $r^2 + r_3^2$, a^2, b^2, c^2 of four points in a plane. The circle will be real or imaginary according as the value of r^2 is positive or negative.

If C and C_1 cut at right angles $CC_1 = 0$, so that if C be at right angles to three circles C_1, C_2, C_3,

$$C(\alpha C_1 - \beta C_2 + \gamma C_3) = \alpha CC_1 + \beta CC_2 + \gamma CC_3 = 0,$$

and C is therefore also at right angles to every circle of the system $\alpha C_1 + \beta C_2 + \gamma C_3$.

When one of the circles C_1, C_2 reduces to a point $r_2 = 0$; and therefore

$$C_1 C_2 = \tfrac{1}{2}(r_1^2 - a_{12}^2),$$

so that the equation $C_1 C_2 = 0$ expresses that the point C lies on the circle C_1. When C_1 reduces to a point

$$C_1 C_2 = -\tfrac{1}{2} a_{12}^2,$$

or the product of two points is half the negative square of their distance.

Whenever the expression $\alpha C_1 + \beta C_2 + \gamma C_3$ becomes a point, it will be a point on the circle cutting C_1, C_2, C_3 at right angles.

Taking the centre of C_1, C_2, C_3 again as the point ABC, the condition that $\alpha C_1 + \beta C_2 + \gamma C_3$ should be a point, is

$$\alpha^2 r_1^2 + \beta^2 r_2^2 + \gamma^2 r_3^2 + \beta\gamma(r_2^2 + r_3^2 - a^2) + \gamma\alpha(r_3^2 + r_1^2 - b^2) + \alpha\beta(r_1^2 + r_2^2 - C^2) = 0,$$

or $$a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta = (\alpha r_1^2 + \beta r_2^2 + \gamma r_3^2)(\alpha + \beta + \gamma).$$

This is therefore the equation in areal coordinates to the circle cutting the three circles C_1, C_2, C_3 at right angles. When the circles reduce to the points A, B, C the equation reduces to the well-known equation to the circumscribing circle

$$a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta = 0.$$

The quantity $C_3 - C_2$ has been seen to represent a line; its square is

$$C_3^2 + C_2^2 - 2C_3C_2 = r_2^2 + r_3^2 - (r_2^2 + r_3^2 - a^2) = a^2.$$

We may call a the magnitude of the line; it is equal to the distance from the centre. It may be considered to be directed upwards when x is taken from left to right. Let L (fig. 6) be the point where this line meets BC, D the middle point of BC, AH the perpendicular from A or BC, D the middle point of BC, then

$$C_1(C_3 - C_2) = \tfrac{1}{2}(r_1^2 + r_3^2 - b^2) - \tfrac{1}{2}(r_1^2 + r_2^2 - c^2)$$
$$= \tfrac{1}{2}(r_3^2 - r_2^2) + \tfrac{1}{2}(c^2 - b^2) = \tfrac{1}{2}(CL^2 - BL^2) + \tfrac{1}{2}(BH^2 - CH^2)$$
$$= a(LD + DH) = aLH = ah,$$

if h be the distance of A from the line.

Thus the product of a circle into a straight line is the product of the magnitude of the line into the distance of the centre of the circle from the line.

If D be centre of another circle C_4 of radius r_4, and $AD = a'$, while h' is the distance of D from the $C_3 - C_2$ and θ the angle AD makes with BC or the line $C_2 - C_1$ makes with $C_3 - C_2$, then

$$(C_4 - C_1)(C_3 - C_2) = a(h' - h) = aa' \cos\theta,$$

or the product of two lines is the product of their lengths into the cosines of the angle between them. This is identical with the inner multiplication of lines in the "Ausdehnungslehre," or the scalar multiplications in quaternions.

Suppose B', C (fig. 7) to be points directly above B, C and centres of circles C_2', C_3' with r_2, r_3 for radii, $C_3' - C_2'$ is a line drawn through a point r' perpendicular to $B'C'$, and is a continuation of the line $C_3 - C_2$.

But $C_2 + C_3' = C_2' + C_3$, since both are circles taken twice over, with O the middle point of BC' for centre, and

$$\tfrac{1}{2}\sqrt{\{r_2^2 + r_3^2 + (r_2^2 + r_3^2 - BC^2)\}}$$

for radius.

Therefore $C_3' - C_2' = C_3 - C_2$, or a line can be transferred anywhere along its own length without being altered.

To find the sum of two intersecting lines OF, OG we may draw through O lines AOA', BOB' at right angles to OF, OG respectively, and make $AO = OA' = \frac{1}{2}OF$, $BO = OB' = OG$, and place equal circles at A, A' B, B'.

Let D, D' (fig. 8) be the middle points of AB, $A'B'$. Then

$$\text{line } OF = \text{circle at } A' - \text{circle at } A,$$

$$\text{line } OG = \text{circle at } B' - \text{circle at } B,$$

$$\text{line } OH + \text{line } OG = \text{circle at } A' + \text{circle at } B' - \text{circle at } A - \text{circle at } B$$

$$= 2\,(\text{circle at } D' - \text{circle at } D)$$

$$= \text{a line perpendicular to } DD' \text{ and of length } 2DD'$$

$$= \text{the line } OH, \text{ if } OH \text{ be the diagonal of the parallelogram with } OF, OG \text{ for sides.}$$

To find the sum of two parallel lines of magnitudes a, b drawn at L, M perpendicular to AB.

Make $AL = LB = \frac{1}{2}a$, $AM = MC = \frac{1}{2}b$ (fig. 9).

Then the line at L is the difference of the points B and A, and the line at M is the difference of the points C and A,

$$\text{line at } L + \text{line at } M = B - A + C - A = B + C - 2A$$

$$= 2\,[\text{circle of radius } \sqrt{\{-\tfrac{1}{4}(a-b)^2\}} \text{ at } D \text{ middle point of } BC-A]$$

$$= \text{line at } N \text{ of magnitude } 2AD,$$

where N is a point such that

$$AN^2 - DN^2 = \tfrac{1}{4}(a-b)^2.$$

Now $2AD = a + b$, or the magnitude of the sum of two parallel lines is the sum of their magnitudes.

Let $MN = y$, $NL = x$,

$$LM = \tfrac{1}{2}(a-b) = CD,$$

$$AL = DM = \tfrac{1}{2}a, \quad AM = DL = \tfrac{1}{2}b,$$

$$2AN^2 = (\tfrac{1}{2}a - x)^2 + (\tfrac{1}{2}b + y)^2,$$

$$2DN^2 = (\tfrac{1}{2}b + x)^2 + (\tfrac{1}{2}a - y)^2.$$

Thus the equation $AN^2 - DN^2 = \frac{1}{4}(a-b)^2$ gives

$$(a+b)(y-x) = \tfrac{1}{2}(a-b)^2;$$

also

$$y + x = \tfrac{1}{2}(a-b),$$

therefore $$y = \frac{a-b}{2(a+b)}\,a, \quad x = \frac{a-b}{2(a+b)}\,b,$$

or $$ax = by.$$

Thus the sum of two parallel lines divides the perpendicular, in the inverse proportion of their magnitudes. The addition of lines (Linientheile) in the Ausdehnungslehre is seen to be included in the more general addition of circles.

When the parallels are equal and of opposite sign, the formula gives a small line at an infinite distance. Thus the difference of equal parallel lines is a new kind of quantity, and it is shewn, as in statics, that this difference is proportional to the product of the length of the lines into the distance between them.

Call Θ the difference of two unit parallel lines at a unit distance; then the difference of two lines of magnitude a at a distance l will be $al\Theta$.

Let $AB = a$, $LM = l$; place circles C_2', C_2 at B, whose radii are r'_2, r_2, and a circle C_1 at A.

Let $C_2 - C_1 =$ line of magnitude a at L,

$C_2' - C_1 =$ line of magnitude a at M,

then $C_2' - C_2 =$ difference of lines at M and L

$= al\Theta$,

if the positive direction of Θ be taken to be that of the hands of a watch. Now

$$BL^2 - AL^2 = r_2^2 - r_1^2, \quad BM^2 - AM^2 = r'^2_2 - r'^2;$$

therefore $$\begin{aligned} r'^2_2 - r_2^2 &= (BM^2 - AM^2) - (BL^2 - AL^2) \\ &= a\{(BM - AM) - (BL - AL)\} \\ &= 2al, \end{aligned}$$

so that $$C_2' - C_2 = \tfrac{1}{2}(r'^2_2 - r_2^2)\,\Theta;$$

or the difference of two concentric circles is one-half the difference of the squares of their centres multiplied into Θ.

In particular, if O be the centre of a circle of radius r,

$$C = O + \tfrac{1}{2}r^2\Theta.$$

Since

$$C_1C_2' = \tfrac{1}{2}(r_1^2 + r'^2_2 - a^2), \quad CC_2 = \tfrac{1}{2}(r_1^2 + r_2^2 - a^2),$$

and $$C_1(C_2' - C_2) = \tfrac{1}{2}(r'^2_2 - r_2^2)\,C\Theta;$$

therefore $$C_1\Theta = 1,$$

or the product of any unit circle into Θ is 1.

Also $\frac{1}{2}(r'^2_2 - r_2^2)\,\Theta^2 = \Theta C'_2 - \Theta C_2 = 1 - 1 = 0,$

or $$\Theta^2 = 0.$$

Again, if L be any unit line whose distance from B is h,

$$C'_2 L = h, \quad C_2 L = h,$$

$$\Theta L = \frac{h - h}{\frac{1}{2}(r'^2_2 - r^2)} = 0.$$

A circle C of multiplicity μ may be put equal to $\mu C'$, where C' is a unit circle; then

$$\Theta C = \mu \Theta C' = \mu.$$

If h be the distance of its centre from a line L,

$$LC = \mu LC' = \mu h,$$

so that $$h = \frac{LC}{\Theta C}.$$

If r be its radius,

$$C^2 = \mu^2 C'^2 = \mu^2 r^2,$$

and thus $$r^2 = \frac{C^2}{(\Theta C)^2}.$$

When C is a line, then, since $\Theta C = 0$, h and r become infinite; when C is identical with Θ, h becomes indeterminate and r^2 infinite, so that Θ may be considered a circle of infinite radius and indeterminate centre.

Special Systems of Circles.

If two circles C, C' touch, then d the distance between their centres is equal to $r + r'$ the sum of their radii, or $r - r'$ the difference of their radii. Therefore

$$CC' = \tfrac{1}{2}(r^2 + r'^2 - d^2) = \mp rr',$$

according as the circles touch externally or internally.

Hence $(CC')^2 = C^2 C'^2$, and this is the necessary and sufficient condition that the circles C, C' should touch.

When C' reduces to a point, $C'^2 = 0$, and therefore $CC' = 0$, or a circle may be considered to touch as well as to cut at right angles a point that lies on it.

If C touch C', it will touch any circle of the system $\lambda C + \mu C'$, since

$$\{C(\lambda C + \mu C')\}^2 = \lambda^2 (C^2)^2 + 2\lambda\mu C^2 (CC') + \mu^2 (CC')^2$$
$$= C^2 (\lambda^2 C^2 + 2\lambda\mu CC' + \mu^2 C'^2)$$
$$= C^2 (\lambda C + \mu C')^2.$$

In particular, if P be a point on the circle C, $\lambda C + \mu P$ represents any circle touching C at the point P.

If t be the direct tangent to the circles C, C' (which do not touch),

$$t^2 + (r - r')^2 = d^2,$$
$$t^2 = 2rr' - (d^2 - r^2 - r'^2) = 2\{+\sqrt{(C^2 C'^2)} - CC'\}.$$

If t' be the transverse tangent,

$$t'^2 = 2\{-\sqrt{(C^2 C'^2)} - CC'\}.$$

The equation $CC' = 0$, expressing that C cuts C' at right angles, is linear. Therefore all the circles cutting a given circle at right angles, may be derived from any three of them. For if C_1, C_2, C_3 be any three, $\alpha C_1 + \beta C_2 + \gamma C_3$ will also cut the given circle at right angle, and the number of circles included in this expression is doubly infinite. If C_1, C_2, C_3 pass through a point P, then

$$PC_1 = 0,\ PC_2 = 0,\ PC_3 = 0.$$

But since $P^2 = 0$, P itself satisfies the equation $PC = 0$, and therefore P can be derived from C_1, C_2, C_3. Put

$$P = \alpha C_1 + \beta C_2 + \gamma C_3.$$

Multiply by C_1, C_2, C_3, and eliminate α, β, γ. We have as the condition, that three circles should pass through a point

$$\begin{vmatrix} C_1^2, & C_1C_3, & C_1C_3 \\ C_1C_2, & C_2^2, & C_2C_3 \\ C_1C_3, & C_2C_3, & C_3^2 \end{vmatrix} = 0.$$

It is easily seen, conversely, that if this condition holds, the three circles must pass through a point. If θ, ϕ, χ be the angles the circles make with one another, the condition gives

$$r_1^2 r_2^2 r_3^2 \begin{vmatrix} 1, & \cos\chi, & \cos\phi \\ \cos\chi, & 1, & \cos\theta \\ \cos\phi, & \cos\theta, & 1 \end{vmatrix} = 0,$$

or, $\theta \pm \phi \pm \chi =$ some multiple π, as ought to be the case.

When four circles C_1, C_2, C_3, C_4 cut the same circle at right angles they must be connected by a linear relation.

Therefore $0 = \alpha C_1 + \beta C_2 + \gamma C_3 + \delta C_4$.

Multiplying by C_1, C_2, C_3, C_4, and eliminating α, β, γ, δ,

$$\begin{vmatrix} C_1^2, & C_1C_2, & C_1C_3, & C_1C_{11} \\ C_1C_2, & C_2^2, & C_2C_3, & C_2C_{11} \\ C_1C_3, & C_2C_3, & C_3^2, & C_3C_{11} \\ C_1C_4, & C_2C_4, & C_3C_4, & C_4^2 \end{vmatrix} = 0.$$

If the circles reduce to points, and d_{12}, d_{13}, &c. be the mutual distances, the condition that four points should lie on a circle, becomes

$$\begin{vmatrix} 0, & d_{12}^2, & d_{13}^2, & d_{14}^2 \\ d_{11}^2, & 0, & d_{13}^2, & d_{14}^2 \\ d_{13}^2, & d_{23}^2, & 0, & d_{34}^2 \\ d_{14}^2, & d_{24}^2, & d_{34}^2, & 0 \end{vmatrix} = 0.$$

Hence, $d_{12}d_{34} + d_{14}d_{13} = d_{13}d_{14}$, and we have the well-known theorem, that the sum of the products of the sides of a quadrilateral inscribed in a circle is equal to the product of its diagonals.

Between any five circles C, C_1, C_2, C_3, C_4, the following relation must hold:

$$\begin{vmatrix} C^2, & CC_1, & CC_2, & CC_3, & CC_4 \\ C_1C, & C_1^2, & C_1C_2, & C_1C_3, & C_1C_4 \\ C_2C, & C_2C_1, & C_2^2, & C_2C_3, & C_2C_4 \\ C_3C, & C_3C_1, & C_3C_2, & C_3^2, & C_3C_4 \\ C_4C, & C_4C_1, & C_4C_2, & C_4C_3, & C_4^2 \end{vmatrix} = 0.$$

Multiply the first column by CC_1, the second by C^2, and subtract; again, the first by CC_2, the third by C^2 and subtract, and so on; divide by $(C^2)^4$, then

$$\begin{vmatrix} C^2C_1^2 - (CC_1)^2, & C^2C_1C_2 - CC_1CC_2, & C^2C_1C_3 - CC_1CC_3, & C^2C_1C_4 - CC_1CC_4 \\ C^2C_1C_2 - CC_1CC_2, & C^2C_2^2 - (CC_2)^2, & C^2C_2C_3 - CC_2CC_3, & C^2C_2C_4 - CC_2CC_4 \\ C^2C_1C_3 - CC_1CC_3, & C^2C_2C_3 - CC_2CC_3, & C^2C_3^2 - (CC_3)^2, & C^2C_3C_4 - CC_3CC_4 \\ C^2C_1C_4 - CC_1CC_4, & C^2C_2C_4 - CC_2CC_4, & C^2C_3C_4 - CC_3CC_4, & C^2C_4^2 - (CC_4)^2 \end{vmatrix} = 0.$$

Now if C touch the circles C_1, C_2, C_3, C_4,

$C^2C_1^2=(CC_1)^2$, $C^2C_2^2=(CC_2)^2$, $C^2C_3^2=(CC_3)^2$, $C^2C_4^2=(CC_4)^2$.

Let t_{12} be the direct tangent to C_1C_2, t'_{12} the transverse. Since

$$CC_1=\mp\sqrt{(C^2C_1^2)},\quad CC_2=\mp\sqrt{(C_1^2C_2^2)},$$

as the circles touch externally or internally,

$$CC_1CC_2=\pm C^2\sqrt{(C_1^2C_2^2)},$$

and $$CC_1CC_2-C^2C_1C_2=C^2\{\pm\sqrt{(C_1^2C_2^2)} - C_1C_2\}=\tfrac{1}{2}C^2t_{12}^2 \text{ or } \tfrac{1}{2}C^2t_{12}'^2,$$

according as C touches C_1, C_2 both externally or internally, or, on the other hand, one externally and the other internally.

When all the contacts are external, we have on substituting

$$\begin{vmatrix} 0, & t_{12}^2, & t_{13}^2, & t_{14}^2 \\ t_{12}^2, & 0, & t_{23}^2, & t_{24}^2 \\ t_{13}^2, & t_{23}^2, & 0, & t_{34}^2 \\ t_{14}^2, & t_{24}^2, & t_{34}^2, & 0 \end{vmatrix}=0,$$

and therefore $$t_{12}t_{34}+t_{23}t_{14}=t_{13}t_{24}.$$

This is Dr. Casey's extension of the theorem connecting four points on a circle.

If A, B, C be the corners of a right-angled triangle, a, b, c the lengths of its sides, then since the line $A-B$ will be at right angles to $A-C$,

$$(A-B)(A-C)=0,$$

or

$$BC=AB+AC,$$

$$a^2=t+c^2,$$

the theorem of Pythagoras.

If α be the angles at A,

$$(A-B)(A-C)=bc\cos\alpha,$$

or

$$\tfrac{1}{2}(a^2+b^2+c^2)=bc\cos\alpha,$$

$$a^2=b^2+c^2-2bc\cos\alpha.$$

All the lines and circles connected with a triangle can be expressed in terms of the three points A, B, C and the circle Θ. Thus, the perpendiculars through the middle points of the sides are $C-B$, $A-C$, $B-A$, and since this sum vanishes they meet in a point.

The perpendiculars from A on BC must be of the form $C-B+\mu\Theta$, since it is parallel to $C-D$. But since it passes through A,

$$A\,(C-B+\mu\Theta)=0,$$

or

$$\tfrac{1}{2}\,(c^2-b^2)+\mu=0\,;$$

therefore the perpendicular is

$$C-B+\tfrac{1}{2}\,(b^2-c^2).$$

The other perpendiculars are

$$A-C+\tfrac{1}{2}\,(c^2-a^2)\,\Theta,\quad B-A+\tfrac{1}{2}\,(a^2-b^2)\,\Theta,$$

and, since their sum vanishes, they meet in a point.

To find the circumscribing circle put it

$$xA+yB+zC+w\Theta.$$

Then since it passes through A, we have multiplying by A

$$0=-\tfrac{1}{2}\,(yc^2+zb^2)+w,$$

or

$$yc^2+ab^2=zw\,;$$

similarly

$$za^2+xc^2=2w,$$

$$xb^2+ya^2=2w\,;$$

$$x=w\,\frac{b^2+c^2-a^2}{b^2c^2},$$

since $x+y+z+w=1$, the circle will be

$$\frac{a^2(b^2+c^2-a^2)\,A+b^2\,(c^2+a^2-b^2)\,B+c^2\,(a^2+b^2-c^2)\,C+a^2b^2c^2\Theta}{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4},$$

and its radius is given by

$$R^2=\frac{a^2b^2c^2\,\{2a^2\,(b^2+c^2-a^2)+\ldots-(c^2+a^2-b^2)\,(a^2+b^2-c^2)\ldots\}}{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)^2}\,;$$

therefore

$$R=\frac{abc}{\sqrt{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-c^4)}}\,.$$

The characteristic property of all straight lines is that when multiplied by Θ the product vanishes.

Now Θ so far resembles a point in that $\Theta^2=0$. If then instead of Θ we take any point O (fig. 11) we shall have instead of straight lines circles passing through the point O, and instead of the triangle ABC three such circles BOC,

COA, AOB. Since O corresponds to a point at infinity, the portion BOC of a circle will correspond to parts of BC produced beyond B and C, and the portion BC will correspond to the side itself. We may see how far the properties of such a triangle will be identical with those of ordinary triangles.

Let the lengths of the lines OA, OB, OC, BC, CA, AB be f, g, h, a, b, c respectively. Then $OA = -\frac{1}{2}f^2$.

Since $\Theta A = 1$, Θ is the limit of $-\frac{2}{f^2}$, O when O passes off to infinity.

To the middle point of the side of a rectilinear triangle will correspond a point D on BC, such that the lines joining $DBOC$ to any point on the circle form an harmonic pencil, that is to say, a point D, and that $\frac{DB}{DC} = \frac{OB}{OC}$.

Put $$D = yB + zC + wO,$$

then $$\frac{za^2 + wg^2}{ya^2 + wh^2} = \frac{g^2}{h^2},$$

or $$\frac{z}{g} = \frac{g^2}{h^2},$$

and since $D^2 = 0$, $$a^2yz + w(g^2y + h^2z) = 0,$$

or $$w = -\tfrac{1}{2}\frac{a^2}{y^2}z,$$

therefore $$D = \frac{\frac{B}{g^2} + \frac{C}{h^2} - \frac{1}{2}\frac{a^2}{g^2h^2}O}{\frac{1}{y^2} + \frac{1}{h^2} - \frac{1}{2}\frac{a^2}{g^2h^2}};$$

when the circles are straight lines, this becomes

$$\tfrac{1}{2}(B + C) + \tfrac{1}{4}a^2\Theta.$$

To the perpendicular through the middle point of the side corresponds the circle through O and D perpendicular to BC. This circle is $\frac{B}{g^2} - \frac{C}{h^2}$, for from its form it cuts BOC at right angles (since it is a circle derived from B and C), and it passes through O and D, since

$$O\left(\frac{B}{g^2} - \frac{C}{h^2}\right) = 0,$$

$$D\left(\frac{B}{g^2} - \frac{C}{h^2}\right) = 0.$$

Similarly, the corresponding circles are $\frac{C}{h^2}-\frac{A}{f^2}$, $\frac{A}{f^2}-\frac{B}{g^2}$, and since their sum vanishes, they meet in a point (not counting O).

To the perpendiculars on the sides of a triangle from the opposite angles correspond the circles through O and A at right angles to BOC and the other two similar circles. The form of this circle must be

$$\frac{B}{g^2}-\frac{C}{h^2}+\mu O,$$

since it passes through O and is perpendicular to BOC; and, since it passes through A,

$$A\left(\frac{B}{g^2}-\frac{C}{h^2}+\mu O\right)=0,$$

or

$$\frac{c^2}{g^2}-\frac{b^2}{h^2}+\mu f^2=0,$$

$$\mu=\frac{b^2g^2-c^2h^2}{f^2g^2h^2}$$

and the circle is

$$\frac{B}{g^2}-\frac{C}{h^2}+\frac{b^2g^2-c^2h^2}{f^2g^2h^2}\,O.$$

The corresponding circles are

$$\frac{C}{h^2}-\frac{A}{f^2}+\frac{c^2h^2-a^2f^2}{f^2g^2h^2}\,O, \qquad \frac{A}{f^2}-\frac{B}{g^2}+\frac{a^2f^2-b^2g^2}{f^2g^2h^2}\,O,$$

and the three meet in a point.

If $D_1=yB+zC+wO$ be the foot of the first of these perpendiculars, then we must have

$$(yg^2-zh^2)\,a^2f^2=(yg^2+zh^2)\,(b^2g^2-c^2h^2)$$

$$yg^2\,(c^2h^2+a^2f^2-b^2g^2)=zh^2\,(a^2f^2+b^2g^2-c^2h^2).$$

Also, since $D_1^2=0$,

$$w=-\frac{a^2y^2}{g^2y+h^2z};$$

hence

$$D_1=\frac{B}{g^2\,(c^2h^2+a^2f^2-b^2g^2)}+\frac{C}{h^2\,(a^2f^2+b^2g^2-c^2h^2)}-\frac{O}{2f^2g^2h^2}.$$

Now it was found on the preceding page that

$$D = \frac{B}{g^2} + \frac{C}{h^2} - \tfrac{1}{2}\frac{a^2}{g^2h^2}\,O,$$

where D corresponds to the middle point of the side of a triangle. Hence

$$\frac{F+D-E}{2\,(c^2h^2+a^2f^2-b^2g^2)} = \frac{B}{g^2\,(c^2h^2+a^2f^2-b^2g^2)} - \frac{O}{4f^2g^2h^2},$$

and $$D_1 = \frac{F+D-E}{2\,(c^2h^2+a^2f^2-b^2g^2)} + \frac{D+E-F}{2\,(a^2f^2+b^2g^2-c^2h^2)}.$$

Therefore the circle containing D, E, F passes through D_1, and in the same way through E_1, F_1.

If H be the point of intersection of the perpendiculars, then

$$H = \frac{A}{f^2\,(b^2g^2+c^2h^2-a^2f^2)} + \frac{B}{g^2\,(c^2h^2+a^2f^2-b^2g^2} + \frac{C}{h^2\,(a^2f^2+b^2g^2-c^2h^2)} + \mu O,$$

where μ is to be determined. For it is easily seen that

$$H\left(\frac{B}{g^2} - \frac{C}{h^2} + \frac{b^2g^2-c^2h^2}{f^2g^2h^2}\,O\right) = 0.$$

Making $H^2 = 0$,

$$\mu\left(\frac{1}{b^2g^2+c^2h^2-a^2f^2}+\ldots\right) + \left(\frac{a^2}{g^2h^2\,(c^2h^2+a^2f^2-b^2g^2)\,(a^2f^2+b^2g^2-c^2h^2)}+\ldots\right) = 0;$$

therefore $$\mu = -\frac{1}{f^2g^2h^2},$$

and $$H = \frac{A}{f^2(b^2g^2+c^2h^2-a^2f^2)} + \frac{B}{g^2(c^2h^2+a^2f^2-b^2g^2)} + \frac{C}{h^2(a^2f^2+b^2g^2-c^2h^2)} - \frac{O}{f^2g^2h^2}.$$

To the middle point of HA in a rectilinear triangle corresponds a point D_2, such that

$$\frac{D_2H}{D_2A} = \frac{OH}{OA}.$$

Hence we find the same way as D was found,

$$D_2 = \frac{H}{OH} + \frac{A}{OA} - \tfrac{1}{2}\frac{HA}{OH.OA}\,O,$$

and we may put

$$D_2 = H.OA + A.OH - \tfrac{1}{2}O.HA.$$

Substituting and neglecting the multiple $-\frac{1}{2}$,

$$D_2 = f^2\left\{\frac{A}{f^2(b^2g^2+c^2h^2-a^2f^2)}+\ldots-\frac{O}{f^2g^2h^2}\right\}$$
$$+A\left(\frac{1}{b^2g^2+c^2h^2-a^2f^2}+\ldots\right)$$
$$-\frac{O}{2g^2h^2}\left(\frac{c^2h^2}{c^2h^2+a^2f^2-b^2g^2}+\frac{b^2g^2}{a^2f^2+b^2g^2-c^2h^2}-1\right).$$

Substituting for A, B, C in terms of D, E, F, O,

$$D_2 = \text{terms in } D, E, F + f^2\left(\frac{3O}{4f^2g^2h^2}-\frac{O}{f^2g^2h^2}\right)$$
$$+\frac{O}{4g^2h^2}(b^2g^2+c^2h^2-a^2f^2)\left(\frac{1}{b^2g^2+c^2h^2-a^2f^2}+\ldots\right)$$
$$-\frac{O}{2g^2h^2}\left(\frac{c^2h^2}{c^2h^2+a^2f^2-b^2g^2}+\frac{b^2g^2}{a^2f^2+b^2g^2-c^2h^2}-1\right)$$

$$= \text{terms in } D, E, F,$$
$$+\frac{O}{4g^2h^2}\left\{\frac{b^2g^2+c^2h^2-a^2f^2}{c^2h^2+a^2f^2-b^2g^2}+\frac{b^2g^2+c^2h^2-a^2f^2}{a^2f^2+b^2g^2-c^2h^2}\right.$$
$$\left.-\frac{2a^2f^2(b^2g^2+c^2h^2-a^2f^2)}{(c^2h^2+a^2f^2-b^2g^2)(a^2f^2+b^2g^2-c^2h^2)}\right\}$$
$$= \text{terms in } D, E, F.$$

Therefore the circle that passes through D, E, F, D_1, E_1, F_1, passes also through D_2, E_2, F_2. This circle may be called the nine-point circle of three circles meeting in a point.

Let I, I_1, I_2, I_3 be the circles touching BOC, COA, AOB. We may call these latter circles, for shortness, C_1, C_2, C_3.

Put
$$I = xA + yB + zC + wO,$$
$$C_1 = \alpha_1 A + \beta_1 B + \gamma_1 C + \delta_1 O.$$

Then remembering that

$$C_1B=0,\ C_1C=0,\ C_1O=0,$$

the equation $$(C_1I)^2=C_1^2I^2,$$

becomes $$x^2(C_1A)^2=\alpha_1C_1A_1I^2,$$

or, if r be the radius of I,

$$x^2=\frac{\alpha_1}{C_1A}r^2.$$

Similarly $$y^2=\frac{\beta_2}{C_2B}r^2,$$

$$z^2=\frac{\gamma_2}{C_3C}r^2.$$

If I touch all the circles internally, then C_1I, C_2I, C_3I will all be positive, and therefore x, y, z must all be negative, or, what comes to the same thing, all positive. Now we have

$$\begin{aligned}
-2C_1A &= \beta_1c^2+\gamma_1b^2+\delta_1f^2,\\
O &= \alpha_1c^2+\quad+\gamma_1a^2+\delta_3g^2,\\
O &= \alpha_1b^3+\beta_1a^2\quad+\delta_1h^2,\\
O &= \alpha_1f^2+\beta_1g^2+\gamma_1h^2\quad.
\end{aligned}$$

Therefore
$$-\tfrac{1}{2}\frac{\alpha_1}{C_1A}=\frac{\begin{vmatrix} 0, & a^2, & g^2\\ a^2, & 0, & h^2\\ g^2, & h^2, & 0\end{vmatrix}}{\begin{vmatrix} 0, & c^2, & b^2, & f^2\\ c^2, & 0, & a^2, & g^2\\ b^2, & a^2, & 0, & h^2\\ f^2, & g^2, & h^2, & 0\end{vmatrix}},$$

or
$$\frac{\alpha_1}{C_1A}=\frac{4a^2g^2h^2}{2b^2g^2c^2h^2+2c^2h^2a^2f^2+2a^2f^2b^2g^2-a^4f^4-b^4g^4-c^4h^4}$$

$$=\frac{4a^2g^2h^2}{\Delta^2}\ \text{say};$$

therefore
$$x^2=\frac{2agh}{\Delta}r,\ y=\frac{2bhf}{\Delta}r,\ z=\frac{2cfg}{\Delta}r.$$

Squaring $I = xA + yB + zC + wO$

$$-r^2 = a^2yz + b^2zx + c^2xy + w(f^2x + g^2y + h^2z)$$

$$= \frac{4abcfgh}{\Delta^2} r^2(af + bg + ch) + \frac{2fgh}{\Delta} wr(af + bg + ch),$$

$$w = -\frac{\Delta}{fgh(af + bg + ch)} r - \frac{2abc}{\Delta} r.$$

Hence

$$\tfrac{1}{2}I = \frac{2agh}{\Delta} A + \frac{2bhf}{\Delta} B + \frac{2cfg}{\Delta} C - \left\{ \frac{\Delta}{2fgh(af + hg + ch)} + \frac{2abc}{\Delta} \right\} O.$$

For I_1, one of the roots must be taken negative, and thus

$$\frac{I_1}{r_1} = -\frac{2agh}{\Delta} A + \frac{2bhf}{\Delta} B + \frac{2cfg}{\Delta} C - \left\{ \frac{\Delta}{fgh(-af + bg + ch)} - \frac{2abc}{\Delta} \right\} O,$$

and the expressions for I_2, I_3 are similarly.

Now $A = \text{terms in } D, E, F + \frac{O}{4g^2h^2}(b^2g^2 + c^2h^2 - a^2f^2)$;

therefore $\frac{I}{r} = \text{terms in } D, E, F$

$$+ \frac{O}{2\Delta fgh} \left\{ af(b^2g^2 + c^2h^2 - a^2f^2) + \ldots - \frac{\Delta^2}{af + bg + ch} - 4abcfgh \right\}$$

$$= \text{terms in } D, E, F - \frac{abc}{\Delta} O,$$

from $\dfrac{\Delta^2}{af + bg + ch} = af(b^2g^2 + c^2h^2 - a^2f^2) + \ldots - 2abcfgh.$

If N be the nine-point circle

$$NI = -\frac{abc}{\Delta} rNO,$$

and the circles N, I will touch if

$$(NI)^2 = N^2I^2,$$

or, if $$\frac{a^2b^2c^2}{\Delta^2} = \frac{N^2}{(NO)^2} = \frac{s}{NO},$$

when $$N = pD + qE + rF + sO.$$

Now we obtain $\frac{s}{NO}$ from the expression for $\frac{\alpha_1}{C_1 A}$ by writing for

$a^2, g^2, h^2, f^2, b^2, c^2,$

$-2DE, -2EF, -2FD, -2OF, -2OD, -2OE.$

But $$-2EF = -2\left(\frac{C}{h^2}+\frac{A}{f^2}-\frac{1}{2}\frac{O}{h^2f^2}\right)\left(\frac{A}{f^2}+\frac{B}{g^2}-\frac{1}{2}\frac{O}{f^2g^2}\right)$$

$$= \frac{a^2}{g^2h^2},$$

and $$-2OF = 2.$$

Therefore, substituting

$$\frac{s}{NO} = \frac{4\,\frac{a^2}{g^2h^2}\,\frac{b^2}{h^2f^2}\,\frac{c^2}{f^2g^2}}{2\,\frac{4b^2c^2}{h^2f^2f^2g^2}+2\,\frac{4c^2a^2}{f^2g^2g^2h^2}+2\,\frac{4a^2b^2}{g^2h^2h^2f^2}-\frac{4a^4}{g^4h^4}-\frac{4b^4}{h^4f^4}-\frac{4c^4}{f^4g^4}}$$

$$= \frac{a^2b^2c^2}{\Delta^2}.$$

Hence the circles N and I touch.

We obtain $\frac{I_1}{r_1}$ from $\frac{I}{r}$ by changing the sign of a;

therefore $$NI_1 = \frac{abc}{\Delta}\, r_1 NO,$$

and $$(NI_1)^2 = N^2 I_1^2.$$

Thus the nine-point circle of three circles meeting in a point touches the four circles that touch the three circles. Feuerbach's theorem follows at once as a particular case.

To find the circle C_4 that cuts at right angles any three circles C_1, C_2, C_3, put

$$C_4 = \alpha C_1 + \beta C_2 + \gamma C_3 + \delta\Theta,$$

and write for shortness

$$a = C_1^2,\ b = C_2^2,\ c = C_3^2,\ f = C_2C_3,\ g = C_3C_1,\ h = C_1C_2,$$

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

$$A = bc - f^2,\quad B = ca - g^2,\quad C = ab - h^2,\quad F = gh - af,$$

$$G = hf - bg,\quad H = fg - ch.$$

Multiply by C_1, C_2, C_3, C_4; if r be the radius of C_4,

$$-\delta = a\alpha + h\beta + g\gamma,$$
$$-\delta = h\alpha + b\beta + f\gamma,$$
$$-\delta = g\alpha + f\beta + c\gamma,$$
$$r^2 = \alpha + \beta + \gamma = 1.$$

Hence

$$-\frac{1}{r^2}\alpha = \frac{A+H+G}{\Delta} \quad -\frac{1}{r^2}\beta = \frac{H+B+F}{\Delta} \quad -\frac{1}{r^2}\gamma = \frac{G+F+C}{\Delta},$$

$$r^2 = \frac{-\Delta}{A+B+C+2F+2G+2H},$$

$\Delta = 0$ gives $r = 0$ and is the condition already found that the three circles should meet in a point.

$A+B+C+2F+2G+2H = 0$ gives $r = \infty$, and is the condition that their centres should lie on a straight line.

To find the circles I that touch the three circles c_1, c_2, c_3 we have the equations

$$(IC_1)^2 = I^2C_1^2, \quad (IC_2)^2 = I^2C_2^2, \quad (IC_3)^2 = I^2C_3^2.$$

Hence
$$\frac{IC_1}{\sqrt{a}} = \pm\frac{IC_2}{\sqrt{b}} = \pm\frac{IC_3}{\sqrt{c}}.$$

From these equations Dr. Casey's construction follows. For the circle $\frac{C_2}{\sqrt{b}} - \frac{C_3}{\sqrt{c}}$ has its centre on the line joining the centres of C_2, C_3 produced and divides that line externally in the ratio $\sqrt{b} : \sqrt{c}$, that is, in the ratio of the radii of the circles. Hence it is the point of intersection of the direct common tangent or the external centre of similitude. If L be that axis of similitude that contains the external centres of similitude of C_2C_3, C_3C_1, and therefore C_1C_2, we have

$$L\left(\frac{C_2}{\sqrt{b}} - \frac{C_3}{\sqrt{c}}\right) = 0, \quad L\left(\frac{C_3}{\sqrt{c}} - \frac{C_1}{\sqrt{a}}\right) = 0,$$

and therefore
$$L\left(\frac{C_1}{\sqrt{a}} - \frac{C_2}{\sqrt{b}}\right) = 0.$$

But
$$C_4\left(\frac{C_2}{\sqrt{b}} - \frac{C_3}{\sqrt{c}}\right) = 0, \quad C_4\left(\frac{C_3}{\sqrt{c}} - \frac{C_1}{\sqrt{a}}\right) = 0,$$

Fig. 11.

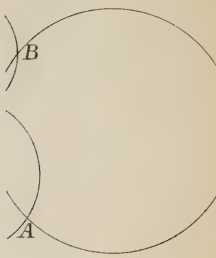
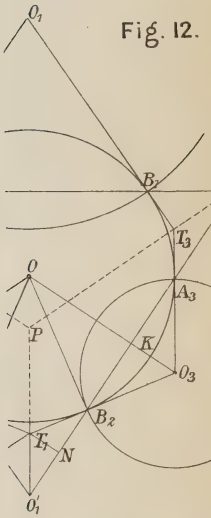


Fig. 12.



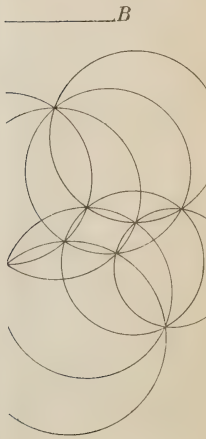


Fig. 1.

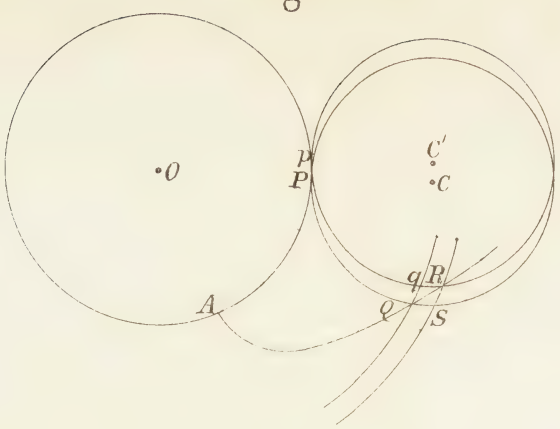


Fig. 3.

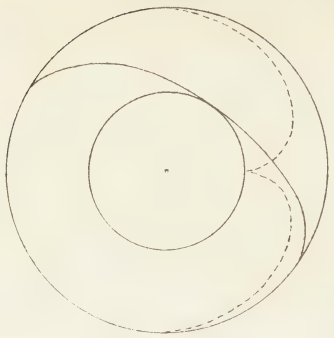


Fig. 11.

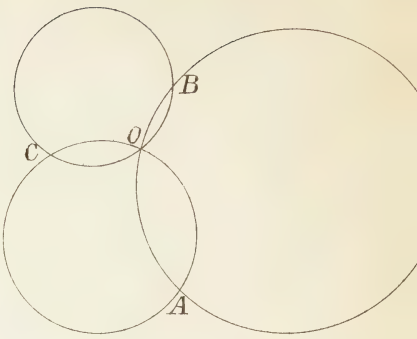


Fig. 3*.

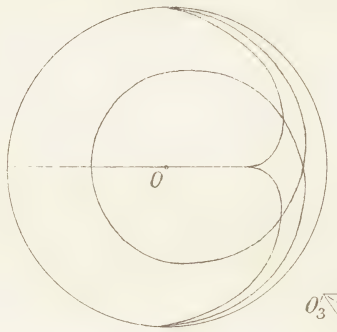


Fig. 2.

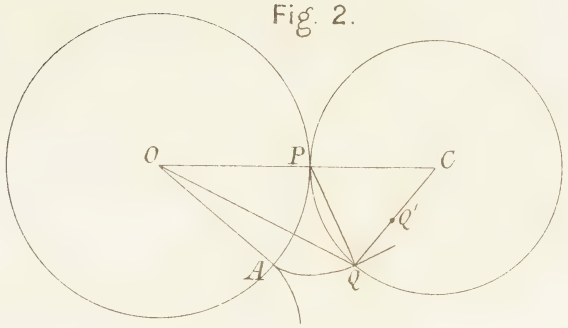


Fig. 12.

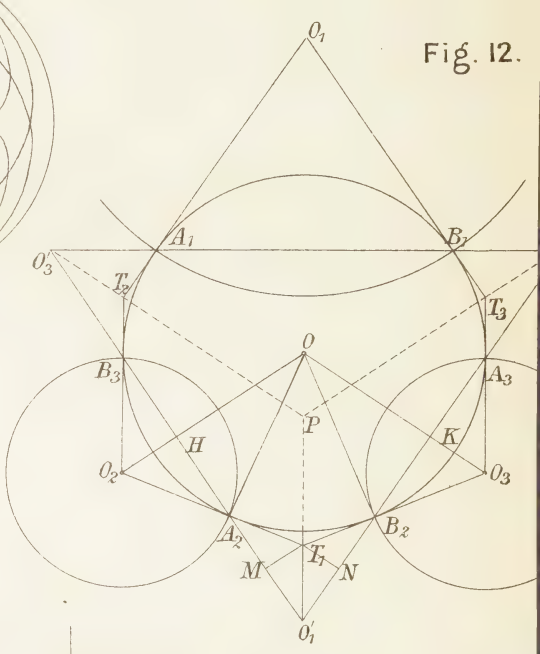


Fig. 4.

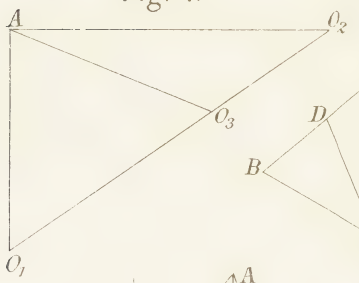


Fig. 8.

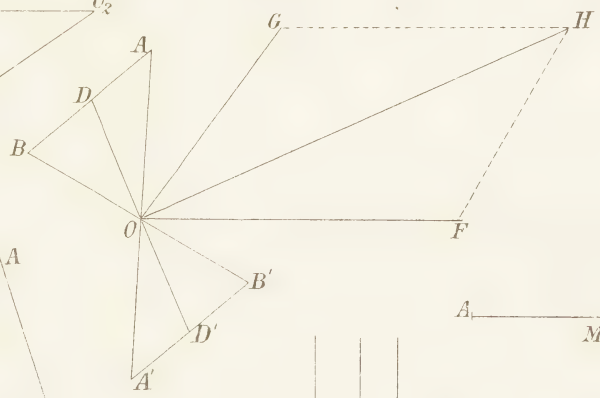


Fig. 6.

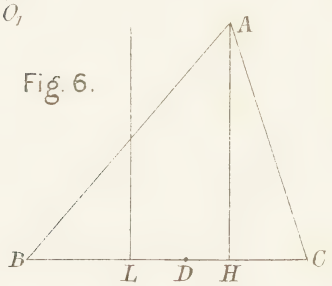


Fig. 10.

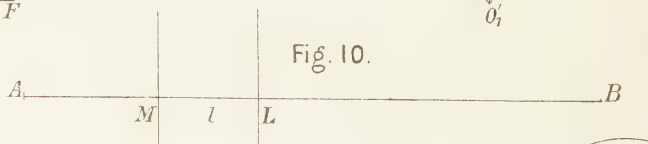


Fig. 9.

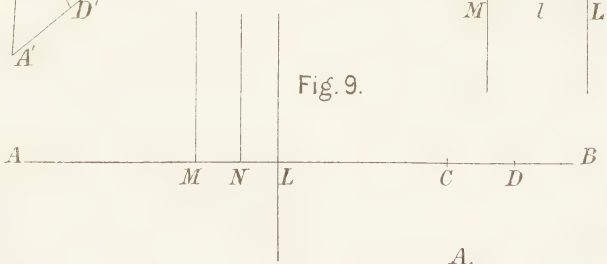


Fig. 13.

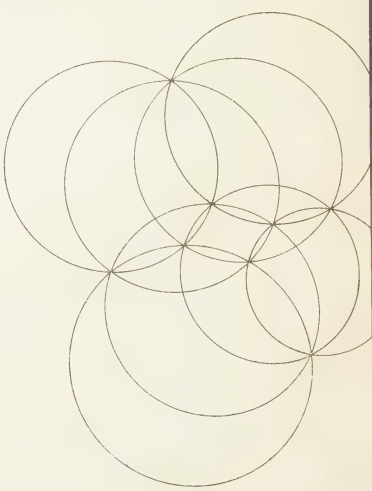


Fig. 7.

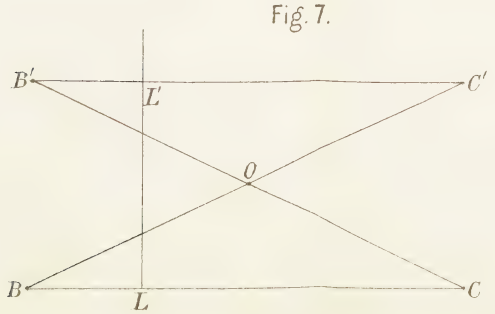
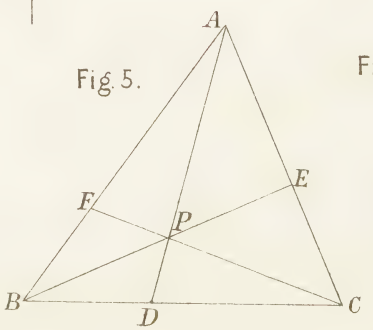


Fig. 5.



Now taking the positive signs in the equations recently written

$$I\left(\frac{C_2}{\sqrt{b}}-\frac{C_3}{\sqrt{c}}\right)=0,\quad I\left(\frac{C_3}{\sqrt{c}}-\frac{C_1}{\sqrt{a}}\right)=0,$$

and therefore I may be derived from L and C_4.

Put $$I=C_4+\mu L,$$

then $$I-C_1=C_4-C_1+\mu L.$$

But $I-C_1$ is the tangent at the point of contact of I with C_1, and the equation shows that this tangent passes through the point of intersection of L with the common chord of C_2, C_4. Similarly, to find the points of contact of I with C_2, C_3 we must draw tangents from the points of intersection of C_2-C_4, C_3-C_4 respectively with L. Since in each case we get two points of contact of six in all, there are two circles I for which IC_1, IC_2, IC_3 are either all positive or all negative. Similarly, there are two circles for each other three chords of similitude. To obtain an expression for I, put

$$I=xC_1+yC_2+zC_3+wC_4;$$

then if ρ be the radins of I,

$$ax+hy+gz=\rho\sqrt{a},$$
$$hx+by+fz=\rho\sqrt{b},$$
$$gx+fy+cz=\rho\sqrt{c}.$$

Hence $$x=\frac{\rho}{\Delta}(A\sqrt{a}+H\sqrt{b}+G\sqrt{c}),$$

and similarily for y and z. Put

$$(A+H+G)\,C_1'=AC_1+HC_2+GC_3,$$
$$(H+B+F)\,C_2'=HC_1+BC_2+FC_3,$$
$$(G+F+C)\,C_3'=GC_1+FC_2+CC_3.$$

Then

$$I=\frac{\rho}{\Delta}\{(A+H+G)\sqrt{(a)}\,C_1'+(H+B+F)\sqrt{(b)}\,C_2'+(G+F+C)\sqrt{(c)}\,C_3'\}+wC_4$$
$$=-\frac{\rho}{\gamma^2}\{\alpha\sqrt{(a)}\,C_2'+\beta\sqrt{(b)}\,C_2'+\gamma\sqrt{(c)}\,C_3'\}+wC_4.$$

To find w, square the equation

$$I=xC_1+yC_2+zC_3+wC_4,$$

then

$$r^2w^2 = \rho^2 - (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)$$

$$= \rho^2 - \rho\,(x\sqrt{a} + y\sqrt{b} + z\sqrt{c})$$

$$= \rho^2\left\{\frac{1 - Aa + Bb + Cc + 2F\sqrt{bc}) + 2G\sqrt{(ca)} + 2H\sqrt{(ab)}}{\Delta}\right\}.$$

If r^2 be positive, or if there be a real circle cutting C_1, C_2, C_3 at right angles, then w must be taken positive, so that the sum of the coefficient of C_1', C_2', C_3', C_4' may be equal to + 1.

Since $C_1'C_2 = 0$, $C_1'C_3 = 0$, C_1' cuts at right angles the three circles C_2, C_3, C_4, and therefore its centre O_1' is the point of intersection of the common chords of (C_2, C_4), (C_3, C_4). If $O_2'O_3'$ be the centres of C_1', C_3', then the triangle $O_1'O_3'O_3'$ is the triangle formed by the lines $C_4 - C_3$, $C_4 - C_1$, $C_4 - C_3'$. The other circles are obtained from I by changing the signs of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$. If I' be the circle touching all three circles externally, then

$$\frac{I}{\rho} + \frac{I'}{\rho'} = \text{a multiple of } C_4.$$

Therefore O is the centre of similitude of I and I'.

Similarly, if I_1 touch C_1 externally and C_2, C_3 internally, while I_1' touch C_1 internally, C_2, C_3 externally, and ρ_1, ρ_1' be their radii,

$$\frac{I_1}{\rho_1} + \frac{I_1'}{\rho_1'} = \text{a different multiple of } C_4.$$

Hence O is the internal centre of similitude of I, I_1', and the eight circles can be arranged in pairs so that O is the internal centre of similitude of each pair. Again

$$\frac{I'}{\rho'} - \frac{I}{\rho} = 2\,\frac{\alpha\sqrt{(a)}\,C_1' + \beta\sqrt{(b)}\,C_2' + \gamma\sqrt{(c)}\,C_3'}{r^2},$$

so that if P be external centre of similitude of $I'I$, the areal coordinates of P referred to the triangle $O_1'O_2'O_3'$ will be $\alpha\sqrt{a}$, $\beta\sqrt{b}$, $\gamma\sqrt{c}$. Also

$$\frac{1}{\rho'} - \frac{1}{\rho} = 2\,\frac{\alpha\sqrt{a} + \beta\sqrt{b} + \gamma\sqrt{c}}{r^2};$$

and since, in the same way,

$$\frac{1}{\rho_1'} - \frac{1}{\rho_1} = 2\,\frac{-\alpha\sqrt{a} + \beta\sqrt{b} + \gamma\sqrt{c}}{r^2},$$

$$\frac{1}{\rho_2'} - \frac{1}{\rho_2} = 2\,\frac{\alpha\sqrt{a} - \beta\sqrt{b} + \gamma\sqrt{c}}{r^2},$$

$$\frac{1}{\rho_3'} - \frac{1}{\rho_3} = 2\,\frac{\alpha\sqrt{a} + \beta\sqrt{b} - \gamma\sqrt{c}}{r^2},$$

therefore $$\frac{1}{\rho'} - \frac{1}{\rho} = \frac{1}{\rho_1'} - \frac{1}{\rho_1} + \frac{1}{\rho_2'} - \frac{1}{\rho_2} + \frac{1}{\rho_3'} - \frac{1}{\rho_2}.$$

If the circles become straight lines, ρ, ρ_1, ρ_2, ρ_3 become infinite, and we have the well-known theorem that the reciprocal of the radius of the inscribed circle is equal to the sum of the reciprocals of the radii of the escribed circles.

We may write the above equation

$$\frac{1}{\rho'} + \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{1}{\rho} + \frac{1}{\rho_1'} + \frac{1}{\rho_2'} + \frac{1}{\rho_3'},$$

or, in other words, the sum of the reciprocals of the radii of four of the circles touching three given circles (namely, of the circle touching all externally and of the three circles touching one externally) is equal to the sum of the reciprocals of the radii of the other four circles (namely, of the circle touching all internally and the three circles touching one internally, two externally).

To find the point P (fig. 12) we may draw radii O_1A_1, O_1B_1; O_2A_2, O_2B_2; O_3A_3, O_3B_3 of the three circles C_1, C_2, C_3 to the points where C_4 meets these circle and joins O'_1, O'_2, O'_3 to the points T_1, T_2, T_3 where these radii intersect. Then P will be the intersection O'_1T_1, O'_2T_2, O'_3T_3.

For if T_1M, T_1N be perpendicular on $O'_1O'_3$,

$$\frac{T_1M}{T_1N} = \frac{\dfrac{T_1M}{T_1A_2}}{\dfrac{T_1N}{T_1B_2}} = \frac{\dfrac{O_2A_2}{O_2O}}{\dfrac{O_3B_3}{O_3O}} = \frac{O_2A_2 \,.\, OH}{O_3B_3 \,.\, OK},$$

since $$OH \,.\, OO_2 = OK \,.\, OO_3.$$

Therefore

$$\frac{M_1M \,.\, O'_3O'_1}{T_1N \,.\, O'_1O'_2} = \frac{\beta\sqrt{b}}{\gamma\sqrt{c}},$$

and the areal coordinates of P are proportional to $\alpha\sqrt{a}$, $\beta\sqrt{b}$, $\gamma\sqrt{c}$.

It must be remembered that the areal coordinates of O are α, β, γ when it is referred to the triangle $O'_1O'_2O'_3$ as well as when it is referred to $O_1O_2O_3$, since

$$(A+H+G)C_1'+(H+B+F)C_2'+(G+F+C)C_3'$$
$$=(A+H+G)C_1+(H+B+F)C_2+(G+F+C)C_3.$$

We may use the values of I, &c., to prove Dr. Hart's theorem that there is a circle J touching the four circles I, I_1, I_2, I_3.

Put

$$\delta^2=1-\frac{Aa+Bb+Cc+2F\sqrt{(bc)}+2G\sqrt{(ca)}+2H\sqrt{(ab)}}{\Delta}$$
$$=\frac{abc+2fgh-af^2-bg^2-ch^2-\{3abc-af^2-bg^2-ch^2+2(gh-af)\sqrt{(bc)}+\ldots\}}{\Delta}$$
$$=\frac{2\{f-\sqrt{(bc)}\}\{g-\sqrt{(ca)}\}\{b-\sqrt{(ab)}\}}{\Delta};$$

and, similarly,

$$\delta_1^2=\frac{2\{f-\sqrt{(bc)}\}\{g+\sqrt{(ca)}\}\{h+\sqrt{(ab)}\}}{\Delta};$$

where the sign of $\sqrt{a}$ is changed

$$\delta_2^2=\frac{2\{f+\sqrt{(bc)}\}\{g-\sqrt{(ca)}\}\{h+\sqrt{(ab)}\}}{\Delta},$$

$$\delta_3^2=\frac{2\{f+\sqrt{(bc)}\}\{g+\sqrt{(ca)}\}\{h-\sqrt{(ab)}\}}{\Delta}.$$

Then
$$\frac{\delta_2^2\delta_3^2}{\delta^2\delta_1^2}=\frac{\{f+\sqrt{(bc)}\}^2}{\{f-\sqrt{(bc)}\}^2}.$$

As δ, δ_1, δ_2, δ_3 are to be taken positive, and $\sqrt{(bc)}$ is greater than f,

$$\frac{\delta_2\delta_3}{\delta\delta_1}=\frac{\sqrt{(bc)}+f}{\sqrt{(bc)}-f},$$

and

$$\frac{f}{\sqrt{(bc)}}=\frac{\delta_2\delta_3-\delta\delta_1}{\delta_2\delta_3+\delta\delta_1},\quad \frac{g}{\sqrt{(ca)}}=\frac{\delta_3\delta_1-\delta\delta_2}{\delta_3\delta_1+\delta\delta_2},\quad \frac{h}{\sqrt{(ab)}}=\frac{\delta_1\delta_2-\delta\delta_3}{\delta_1\delta_2+\delta\delta_3}.$$

Also, using the value of δ^2, δ_1^2, δ_2^2, δ_3^2 in their original form,

$$\delta^2+\delta_1^2+\delta_2^2+\delta_3^2=4-4\frac{Aa+Bb+Cc}{\Delta},$$

$$\delta^2+\delta_1^2-\delta_2^2-\delta_3^2=\frac{-8F\sqrt{(bc)}}{\Delta},$$

$$\delta^2+\delta_2^2-\delta_3^2-\delta_1^2=\frac{-8G\sqrt{(ca)}}{\Delta},$$

$$\delta^2+\delta_3^2-\delta_1^2-\delta_2^2=\frac{-8H\sqrt{(ab)}}{\Delta}.$$

Let $$\frac{1}{\sigma}J=pC_1+qC_2+sC_3+tC_4;$$

then, if J touches I externally and I_1, I_2, I_3 internally,

$$JI=-\rho\sigma,\quad JI_1=\rho_1\sigma,\quad JI_2=\rho_2\sigma,\quad JI_3=\rho_3\sigma.$$

Now

$$\begin{aligned}\frac{1}{\rho\sigma}JI&=(pC_1+qC_2+sC_3+tC_4)\\&\quad\times\left(-\frac{\alpha\sqrt{a}}{r^2}C_1'-\frac{\beta\sqrt{b}}{r^2}C_2'-\frac{\gamma\sqrt{c}}{r^2}C_3'+\frac{\delta}{r}C_4\right)\\&=p\sqrt{a}+q\sqrt{b}+s\sqrt{c}+tr\delta,\end{aligned}$$

and

$$\frac{1}{\rho_1\sigma}JI_1=-p\sqrt{a}+q\sqrt{b}+s\sqrt{c}+tr\delta_1.$$

Hence
$$\begin{aligned}&-(p\sqrt{a}+q\sqrt{b}+\sqrt{c}+tr\delta)\\&=-p\sqrt{a}+q\sqrt{b}+s\sqrt{c}+tr\delta_1\\&=\ \ p\sqrt{a}-q\sqrt{b}+s\sqrt{c}+tr\delta_2\\&=\ \ p\sqrt{a}+q\sqrt{b}-s\sqrt{c}+tr\delta_3=1.\end{aligned}$$

Therefore $$tr(-\delta+\delta_1+\delta_2+\delta_3)=\ \ 4,$$

and $$tr(\ \ \delta-\delta_1+\delta_2+\delta_3)=-4p\sqrt{a}.$$

Substituting

$$\begin{aligned}\frac{\delta_1+\delta_2+\delta_3-\delta}{\sigma}J=\frac{4C_4}{r}&-\frac{\delta-\delta_1+\delta_2+\delta_3}{\sqrt{a}}C_1\\&-\frac{\delta-\delta_2+\delta_3+\delta_1}{\sqrt{b}}C_2-\frac{\delta-\delta_3+\delta_1+\delta_2}{\sqrt{c}}C_3.\end{aligned}$$

Now $J^2 = \sigma^2$, and therefore squaring

$$(\delta_1 + \delta_2 + \delta_3 - \delta)^2 = 16 + (\delta - \delta_1 + \delta_2 + \delta_3)^2 + (\delta - \delta_2 + \delta_3 + \delta_1)^2$$
$$+ (\delta - \delta_3 + \delta_1 + \delta_2)^2 + 2(\delta - \delta_2 + \delta_3 + \delta_1)(\delta - \delta_3 + \delta_1 + \delta_2)\frac{f}{\sqrt{bc}} + \ldots.$$

This is the necessary and sufficient condition in order that J should touch the four circles I, I_1, I_2, I_3.

We may write it (dividing by 2),

$$\delta^2 + \delta_1^2 + \delta_2^2 + \delta_3^2 + 2(\delta\delta_1 - \delta_2\delta_3) + 2(\delta\delta_2 - \delta_3\delta_1) + 2(\delta\delta_3 - \delta_1\delta_2)$$
$$+ \{\delta^2 + \delta_1^2 - \delta_2^2 - \delta_3^2 + 2(\delta\delta_1 + \delta_2\delta_3)\}\frac{f}{\sqrt{bc}}$$
$$+ \{\delta^2 + \delta_2^2 - \delta_0^2 - \delta_1^2 + 2(\delta\delta_2 + \delta_3\delta_1)\}\frac{g}{\sqrt{ca}}$$
$$+ \{\delta^2 + \delta_3^2 - \delta_1^2 - \delta_2^2 + 2(\delta\delta_3 + \delta_1\delta_2)\}\frac{h}{\sqrt{ab}} + 8 = 0.$$

Substituting

$$4 - 4\frac{Aa + Bb + Cc}{\Delta} - 8\frac{Ff}{\Delta} - 8\frac{Gg}{\Delta} - 8\frac{Hh}{\Delta} + 8 = 0.$$

But this is identically true, since

$$Aa + Hh + Gg = Hh + Bb + Ff = Gg + Ff + Cc = \Delta.$$

Dr. Hart's theorem is thus proved. Since the values of δ, δ_1, δ_2, δ_3 are unaltered by changing the signs of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ simultaneously, we may put

$$\frac{\delta_1 + \delta_2 + \delta_3 - \delta}{\sigma'}J' = \frac{4C_4}{r} + \frac{\delta - \delta_1 + \delta_2 + \delta_1}{\sqrt{a}}C_1$$
$$+ \frac{\delta - \delta_2 + \delta_3 + \delta_1}{\sqrt{b}}C_2 + \frac{\delta - \delta_3 + \delta_1 + \delta_2}{\sqrt{c}}C_3,$$

where σ' is the radius of J', and J' will touch the circles I', I_1', I_2', I_3'.

Also $$\frac{J}{\sigma} + \frac{J'}{\sigma'} = \text{a multiple of } C_4.$$

There are therefore two of Dr. Hart's circles, and their internal centre of similitude is the centre of the circle which cuts the three original circles at right angles.

It is easy to see that the circles I, I_1, I', I_1' are touched by a circle, and so with all the pairs of I, I_1, I_2, I_3, and the

corresponding pairs of I', I_1', I_2', I_3'. Thus, altogether the eight touching circles I are touched four and four by eight new circles.

Let θ_1 be the angle J makes with C_1, then multiplying the expressions for J by $\frac{C_1}{\sqrt{a}}$,

$$-(\delta_1+\delta_2+\delta_3-\delta)\cos\theta_1=(\delta-\delta_1+\delta_2+\delta_3)+(\delta-\delta_2+\delta_3+\delta_1)\frac{\delta_1\delta_2-\delta\delta_3}{\delta_1\delta_2+\delta\delta_3}$$

$$+(\delta-\delta_3+\delta_1+\delta_2)\frac{\delta_3\delta_1-\delta\delta_2}{\delta_3\delta_1+\delta\delta_2}$$

$$=\delta\left\{1+\frac{\delta_1\delta_2-\delta\delta_3}{\delta_1\delta_2+\delta\delta_3}+\frac{\delta_3\delta_1-\delta\delta_2}{\delta_3\delta_1+\delta\delta_2}+\frac{(\delta_1\delta_2-\delta\delta_3)(\delta_3\delta_1-\delta\delta_3)}{(\delta_1\delta_2+\delta\delta_3)(\delta_3\delta_1-\delta\delta_3)}\right\}$$

$$+\delta_1\{-1+(\quad)+(\quad)-(\quad)\}+\delta_2\{1-(\quad)+(\quad)-(\quad)\}$$

$$+\delta_3\{\ 1+(\quad)-(\quad)-(\quad)\}+(\delta_1+\delta_2+\delta_3-\delta)\frac{(\delta_1\delta_2-\delta\delta_3)(\delta_3\delta_1-\delta\delta_2)}{(\delta_1\delta_2+\delta\delta_3)(\delta_3\delta_1-\delta\delta_2)}$$

$$=\frac{4\delta\delta_1^2\delta_2\delta_3-4\delta_1\delta^2\delta_2\delta+4\delta_2\delta\delta_3^2\delta_1+4\delta_3\delta_1\delta_2^2\delta}{(\delta_1\delta_2+\delta\delta_3)(\delta_1\delta_3+\delta\delta_2)}$$

$$+\frac{(\delta_1\delta_2-\delta\delta_3)(\delta_1\delta_3-\delta\delta_2)}{(\delta_1\delta_2+\delta\delta_3)(\delta_1\delta_3-\delta\delta_2)}(\delta_1+\delta_2+\delta-\delta)$$

$$=(\delta_1+\delta_2+\delta_3-\delta)\frac{(\delta_1\delta_2-\delta\delta_3)(\delta_1\delta_3-\delta\delta_2)+4\delta\delta_1\delta_2\delta_3}{(\delta_1\delta_2+\delta\delta_3)(\delta_1\delta_3+\delta\delta_2)}.$$

Disregarding signs,

$$\cos\theta_1=\frac{g}{\sqrt{(ca)}}\frac{h}{\sqrt{(ab)}}+\sqrt{\left\{\left(1-\frac{g^2}{ca}\right)\left(1-\frac{h^2}{ab}\right)\right\}}$$

$$=\cos\alpha_2\cos\alpha_3+\sin\alpha_2\sin\alpha_3$$

$$\theta_1=\alpha_2-\alpha_3 \text{ and similarly } \theta_2=\alpha_3-\alpha_1,\ \theta_3=\alpha_1-\alpha_2,$$

if α_1, α_2, α_3 be the angles the circles make with one another. Hence, Dr. Hart's makes angles with the three given circles which are the differences of the angles these circles make with one another. This is obviously true of the nine-point circle of a triangle, and the theorem would be sufficient to shew that the circle of Feuerbach's theorem is the circle passing through the middle points of the triangle.

To find Dr. Hart's circle will be included in the general problem of finding a circle which will cut three circles at given angles.

There will always be two such circles (except when the angles are right angles), and they will have the centre of the circle cutting all three circles at right angles for their external or internal centre of similitude. The angle they both make with this circle is given by

$$\sin^2\theta_4 = \frac{\left\{\begin{array}{l} Aa\cos^2\theta_1 + Bb\cos^2\theta_2 + Cc\cos^2\theta_3 \\ +\,2F\sqrt{(bc)}\cos\theta_2\cos\theta_3 + 2G\sqrt{(ca)}\cos\theta_3\cos\theta_1 + \ldots \end{array}\right\}}{\Delta}.$$

I' is found if I be known, or I_1' if I_1 be known, &c., by taking L one of the points of intersection of I and C_4. Joining O_2 the centre of I and O the centre of C_4 to L, and making the angle O_2LO equal to OLO'_2. Then O_2' will be the centre of I', and it passes through L. L is, as is seen from the proof of Dr. Casey's method, one of the points of intersection of an axis of similitude with C_4.

The quantity

$$A+B+C+2F+2G+2H=(C_1-C_2)^2(C_1-C_3)^2-\{(C_1-C_2)(C_1-C_3)\}^2$$
$$=4 \text{ times the area of triangle } O_1O_2O_3,$$

and its vanishing is the condition that the centres of the circles should lie in a straight line.

It is known that the circles circumscribing the triangles formed by taking four lines three by three all pass through a point, which point is in fact the focus of the parabola touched by these lines. The generalized theorem for circles would be: if four circles pass through a point, and, taking them three by three, four other circles be drawn round the triangles formed by their points of intersection, these four circles also pass through a point. It may be noticed that there is a symmetry in this theorem which is lost when the circle degenerates into a straight line. There are eight circles and eight points, through each point pass four circles and on each circle lie four points.

To prove it, let O be the point common to the four circles, A, B, C, D these circles, P_{12}, P_{13}, P_{14}, P_{24}, P_{34}, P_{23} their other points of intersection, A', B', C', D' the circles passing through these points of intersection. Then, since O, P_{12}, P_{13}, P_{14} lie on the same circle, we may put

$$O = a_2P_{12} + a_3P_{13} + a_4P_{14},$$

Similarly

$$O = b_2P_{12} + b_3P_{23} + b_4P_{24}.$$
$$O = c_1P_{13} + c_2P_{23} + c_4P_{34},$$
$$O = d_1P_{14} + d_2P_{24} + d_3P_{34}.$$

Determining P_{12}, P_{13}, P_{14} from the last three equations, and substituting in the fourth,

$$O = \left(\frac{a_2}{b_1} + \frac{a_3}{c_1} + \frac{a_4}{d_1}\right) O - \frac{a_2}{b_1}(b_3 P_{23} + b_4 P_{24}) - \frac{a_3}{c_1}(c_2 P_{23} + c_4 P_{34}) - \frac{a_4}{d_1}(d_2 P_{24} + d_3 P_{34}).$$

Now O, P_{23}, P_{34}, P_{42} cannot be on the same circle. Hence their coefficients in this equation must vanish identically.

Therefore

$$1 = \frac{a_2}{b_1} + \frac{a_3}{c_1} + \frac{a_4}{d_1},$$

$$0 = \frac{a_3 c_4}{c_1} + \frac{a_4 d}{d_1},$$

$$0 = \frac{a_4 d_2}{d_1} + \frac{a_2 b_4}{b_1},$$

$$0 = \frac{a_2 b_3}{b_1} + \frac{a_3 c_4}{c_1}.$$

Other equations may be obtained in the same way, but none of them will be independent.

From the first and second of the first set of equations

$$a_3 P_{13} - b_3 P_{23} = (b_1 - a_2) P_{12} + b_4 P_{24} - a_4 P_{14}.$$

Therefore

$$\frac{a_2 b_3}{c_2} P_{12} + a_3 P_{13} - b_3 P_{23} = \left(b_1 - a_2 + \frac{a_2 b_3}{c_2}\right) P_{12} + b_4 P_{24} - a_4 P_{14}$$

$$= -\frac{a_2 b_4}{d_2} P_{12} + b_4 P_{24} - a_4 P_{14}.$$

Hence the circle

$$P_{12} - \frac{c_1}{b_1} P_{13} - \frac{c_2}{a_2} P_{23} = \frac{c_2}{a_2 b_3}\left(\frac{a_2 b_3}{c_2} P_{12} + a_3 P_{13} - b_3 P_{23}\right)$$

it at right angles, not only to the circle D', but also to the circle C'.

Again, combining the first and third equations,

$$a_2\left(P_{12}-\frac{c_1}{b_1}P_{13}-\frac{c_2}{a_2}P_{23}\right)=\left(c_1-\frac{a_2c_1}{b_1}-a_3\right)P_{13}-a_4P_{14}+c_4P_{34}$$

$$=\frac{a_4c_1}{d_1}P_{13}-a_4P_{14}+c_4P_{34};$$

and, combining the second and third,

$$b_1\left(P_{12}-\frac{c_1}{b_1}P_{13}-\frac{c_2}{a_2}P_{23}\right)=\left(c_2-b_3-\frac{b_1c_2}{a_2}\right)P_{23}+c_4P_{34}-b_4P_{24}$$

$$=\frac{b_4c_2}{d_2}P_{23}+c_4P_{34}-b_4P_{24}.$$

Hence the circle $P_{12}-\frac{c_1}{b_1}P_{13}-\frac{c_2}{a_2}P_{23}$ is also at right angles to A', B', and therefore to all four circles A', B', C', D'. Since it has not yet been assumed that O, P_{12}, P_{13}, &c. are points, we may take them to be circles, and then we shall have a symmetrical figure of sixteen circles, in which each circle has four circles at right angles to it, and might be taken as the circle to commence with. When O, P_{12}, P_{13}, &c. are points, then we shall have

$$P_{12}-\frac{c_1}{b_1}P_{13}-\frac{c_2}{a_2}P_{23}$$

a point, or

$$\left(P_{12}-\frac{c_1}{b_1}P_{13}-\frac{c_2}{a_2}P_{23}\right)^2=0,$$

$$\frac{c_1c_2}{b_1a_2}P_{13}P_{23}-\frac{c_1}{b_1}P_{12}P_{13}-\frac{c_2}{a_2}P_{12}P_{23}=0.$$

For, since $$a_4P_{14}=O-a_2P_{12}-a_3P_{13},$$

squaring $$a_2a_3P_{12}P_{13}=a_2OP_{12}+a_3OP_{13},$$

$$P_{12}P_{13}=\frac{1}{a_3}OP_{12}+\frac{1}{a_2}OP_{13}.$$

Similarly $$P_{12}P_{23}=\frac{1}{b_3}OP_{12}+\frac{1}{b_1}OP_{23},$$

$$c_1c_2P_{13}P_{23}=c_1OP_{13}+c_2OP_{23}.$$

Hence

$$\frac{c_1c_2}{b_1a_2}P_{13}P_{23} - \frac{c_1}{b_1}P_{12}P_{13} - \frac{c_2}{a_2}P_{12}P_{23}$$

$$= \frac{c_1}{b_1a_2}OP_{13} + \frac{c_2}{b_1a_2}OP_{23} - \frac{c_1}{b_1a_3}OP_{12} - \frac{c_1}{b_1a_2}OP_{13}$$

$$- \frac{c_2}{a_2b_3}OP_{12} - \frac{c_1}{a_2b_1}OP_{23}$$

$$= 0 \text{ identically.}$$

Thus the theorem is completely proved (fig. 13).

Next take five circles A, B, C, D, E, and let, as before, P_{12}, P_{13}, &c. be their points of intersection; C_{123} the circle passing through P_{12}, P_{13}, P_{23}; O_{1234} the point of intersection of C_{123}, C_{124}, C_{234}, C_{341}, and so on. The points O_{1245}, O_{1253}, O_{1234} lie two by two on the three circles C_{123}, C_{124}, C_{125}. But these circles all pass through the point P_{12}, and the circle A also passes through this point. We may apply, then, the theorem to the circles A, C_{123}, C_{124}, C_{125}, and it will follow that the four circles which circumscribe $(P_{13}, P_{14}, O_{1234})$, $(P_{13}, P_{15}, O_{1235})$, $(P_{14}, P_{15}, O_{1245})$, $(O_{1234}, O_{1235}, O_{1245})$ pass through a point. But the first three circles are C_{134}, C_{135}, C_{145}, and their point of intersection is the point O_{1345}. Hence this point lies on the circle through O_{1234}, O_{1235}, O_{1245}. Similarly, by considering the circles B, C_{123}, C_{124}, C_{125}, it is seen that O_{2345} lies on the same circle. Hence all five points O_{1234}, O_{1235}, O_{1245}, O_{1345}, O_{2345} lie on the same circle. Thus we have a figure consisting of $5 + 10 + 1 = 16$ circles and 16 points, such that each circle contains five points and through each point pass five circles. We might start with any point, P_{12} for example, and the circles through it, and we could construct the whole system, ending with C_{345}. Miquel's theorem is a particular case of this, for when the circles A, B, C, D, E becomes a straight line, O_{1234}, &c., become the foci of the parabola's touching them four and four, and it is proved that these foci lie on a circle. In the case of the circles we may call the final circle C_{12345}. Next take six circles through the point O. Through P_{12} also pass six circles, namely, A, B, C_{123}, C_{124}, C_{125}, C_{126}. Consider the last four of these; their points of intersection are

$$O_{1234},\quad O_{1235},\quad O_{1236},\quad O_{1245},\quad O_{1246},\quad O_{1256},$$

and the circles passing through them are

$$C_{12345},\quad C_{12356},\quad C_{12346},\quad C_{12456}.$$

These circles therefore pass through a point, and it is seen in the same way that C_{23456}, C_{13456} pass through the same point. Thus we have a figure of 32 circles and 32 points, such that each circle contain six points and through each point pass six circles. When the first six points become straight lines we have the theorem that the six Miquel circles obtained from a hexagon by leaving out side after side in turn pass through a point. This may be called a hexagon point. In the case of circles we may write for the point O_{123456}. Take now seven circles, and consider four of the circles through O_{1234}, namely, C_{123}, C_{12345}, C_{12346}, C_{12347}. Their points of intersection are O_{1235}, O_{1236}, O_{1237}, O_{123456}, O_{123457}, O_{123467}, and three of the circles through these points are C_{12356}, C_{13357}, C_{12367}, which intersect in the point O_{123567}. This point therefore lies on the circle through O_{123456}, O_{123457}, O_{123467}, and so it may be shown do the points O_{124567}, O_{134567}, O_{234567}. As a particular case, the seven hexagon points obtained from a heptagon by leaving out side after side lie on a circle, which may be called a heptagon circle. These theorems may obviously be extended indefinitely. Thus eight heptagon circles pass through an octagon point, nine octagon points lie on a nonagon circle, &c. In general, from a polygon of an even number of sides we shall ultimately obtain a point, and from a polygon of an odd number of sides a circle. Also we can have a system of 2^{n-1} circles and 2^{n-1} points, such that each circle contains n points and through each point pass n circles.

To find the relation between the distance a, b, c, d, e, f of any four points on a plane, let A, B, C, D be the points, and put

$$\Theta = \alpha A + \beta B + \gamma C + \delta D,$$

then
$$\alpha + \beta + \gamma + \delta = 0,$$

and
$$\beta c^2 + \gamma b^2 + \delta d^2 = -2,$$
$$\alpha c^2 + \gamma a^2 + \delta e^2 = -2,$$
$$\alpha b^2 + \beta a^2 + \delta f^2 = -2,$$
$$\alpha d^2 + \beta e^2 + \gamma f^2 = -2.$$

Solving from the last four equations, and substituting in the first, we shall have as the condition that the sum of the minors of the determinant

$$\begin{vmatrix} 0, & c^2, & b^2, & d^2 \\ c^2, & 0, & a^2, & e^2 \\ b^2, & a^2, & 0, & f^2 \\ d^2, & e^2, & f^2, & 0 \end{vmatrix}$$

must vanish.

Lastly, if four points be on a circle,

$$\alpha A + \beta B + \gamma C + \delta D = 0.$$

Hence $\alpha A + \beta B = -\gamma C - \delta D =$ a circle with centre O and radius r say, where O is the point of intersection of AB and CD.

But it was seen that when the sum of multiples of two points A, B is equal to a circle of centre O and radius r,

$$\text{distance } OA \times \text{distance } OB = r^2;$$

therefore

$$\text{distance } OA \times \text{distance } OB = \text{distance } OC \times \text{distance } OD.$$

Coordinates of Circles and Orthogonal Systems.

When $$C = xC_1 + yC_2 + zC_3 + wC_4,$$

where C_1, C_2, C_3, C_4 are fixed circles, x, y, z, w may be called the coordinates of C. The most convenient system of coordinates will be when C_1, C_2, C_3, C_4 are mutually at right angles. Then we shall have also

$$(C_1 - C_2)(C_3 - C_4) = 0, \quad (C_1 - C_3)(C_2 - C_4) = 0,$$
$$(C_1 - C_4)(C_2 - C_3) = 0.$$

Hence the line joining any two of the centres is at right angles to the line joining the other two, or the centre of any one circle is the orthocentre of the triangle formed by the three others. Three of these centres must form an acute-angled triangle. We will take these for the centres of the circles C_1, C_2, C_3. Now

$$(C_1 - C_2)(C_1 - C_3) = C_1^2 = (\text{radius of } C_1)^2,$$

but $(C_1 - C_2)(C_1 - C_3)$ is positive, since the triangle formed by the centres of C_1, C_2, C_3 is acute, therefore the radius of C_1 is real. Similarly the radii of C_2, C_3 are real. But $(C_4 - C_1)(C_4 - C_2) = C_4^2$ is negative, since the angle formed by $C_4 - C_1$, $C_4 - C_2$ is obtuse.

Therefore the radius of C_4 is imaginary.

We will put

$$C_1^2 = r^2, \quad C_2^2 = r_2^2, \quad C_3^2 = r_3^2, \quad C_4^2 = -r_4^2.$$

Since $$\Theta C_1 = 1, \quad \Theta C_2 = 1, \quad \Theta C_3 = 1, \quad \Theta C_4 = 1,$$

we must have $$\Theta = \frac{C_1}{r_1^2} + \frac{C_2}{r_2^2} + \frac{C_3}{r_3^2} - \frac{C_4}{r_4^2},$$

and since $\Theta^2 = 0$,

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} - \frac{1}{r_4^2} = 0.$$

It is seen then that three of the quantities r_1, r_2, r_3, r_4 can be chosen arbitrarily, and that when this is done, the fourth is determined, as well as the distances between their centres.

Changing the notation slightly, we will put C_1', C_2', C_3', C_4' for what we have hitherto called C_1, C_2, C_3, C_4, and let

$$C_1 = \frac{1}{r_1} C_1', \quad C_2 = \frac{1}{r_2} C_2', \quad C_3 = \frac{1}{r_3} C_3', \quad C_4 = \frac{1}{r_4} C_4',$$

so that

$$\Theta C_1 = \frac{1}{r_1}, \quad \Theta C_2 - \frac{1}{r_2}, \quad \Theta C_3 = \frac{1}{r_3}, \quad \Theta C_4 = \frac{1}{r_4},$$

$$C_1^2 = 1, \quad C_2^2 = 1, \quad C_3^2 = 1, \quad C_4^2 = -1.$$

If now $C = xC_1 + yC_2 + zC_3 + wC_4$,

x, y, z, w may be called the orthogonal coordinates of C.

The multiplicity of C is

$$\Theta C = \frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3} + \frac{w}{r_4},$$

and its radius is given by

$$r^2 = \frac{C^2}{(\Theta C)^2} = \frac{x^2 + y^2 + z^2 - w^2}{\left(\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3} + \frac{w}{r_4}\right)^2}.$$

If C be a point $x^2 + y^2 + z^2 - w^2 = 0$.

In this case putting

$$P = xC_1 + yC_2 + zC_3 + wC_4,$$

it is easy to see the meaning of x, y, z, w. For

$$x = PC_1 = \frac{PC_1'}{r_1} = -\frac{t_1^2}{2r_1},$$

if t_1 be the tangent from P to C_1.

Similarly $y = -\frac{t_2^2}{2r_2}$, $z = -\frac{t_3^2}{2r_3}$, $w = \frac{t_4^2}{2r_4}$,

and hence $$\frac{t_1^4}{r_1^2} + \frac{t_2^4}{r_2^2} + \frac{t_3^4}{r_3^2} - \frac{t_4^4}{r_4^2} = 0,$$

a relation between the tangents from any point to the four circles.

The general equation of the first degree

$$lx + my + nz - rw = 0$$

represents all the circles cutting at right angles the circle whose coordinates are l, m, n, r. If

$$\begin{aligned} D_1 &= \lambda_1 C_1 + \mu_1 C_2 + \nu_1 C_3 - \rho_1 C_4, \\ D_2 &= \lambda_2 C_1 + \mu_2 C_2 + \nu_2 C_3 - \rho_2 C_4, \\ D_3 &= \lambda_3 C_1 + \mu_3 C_2 + \nu_3 C_3 - \rho_3 C_4, \\ D_4 &= \lambda_4 C_1 + \mu_4 C_2 + \nu_4 C_3 - \rho_4 C_4, \end{aligned}$$

be four new orthogonal circles, such that

$$D_1^2 = 1,\ D_2^2 = 1,\ D_3^2 = 1,\ D_4^2 = -1,$$

$$D_2 D_3 = 0,\ D_3 D_1 = 0,\ D_1 D_2 = 0,\ D_1 D_4 = 0,\ D_2 D_4 = 0,\ D_3 D_4 = 0,$$

that is to say, such that

$$\begin{aligned} \lambda_1^2 + \mu_1^2 + \nu_1^2 - \rho_1^2 &= 1, \\ \lambda_2^2 + \mu_2^2 + \nu_2^2 - \rho_2^2 &= 1, \\ \lambda_3^2 + \mu_3^2 + \nu_3^2 - \rho_3^2 &= 1, \\ \lambda_4^2 + \mu_4^2 + \nu_4^2 - \rho_4^2 &= -1, \\ \lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3 - \rho_2\rho_3 &= 0, \\ \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1 - \rho_3\rho_1 &= 0, \\ \lambda_1\lambda_2 + \mu_1\lambda_2 + \nu_1\nu_2 - \rho_1\rho_2 &= 0, \\ \lambda_1\lambda_4 + \mu_1\mu_4 + \nu_1\nu_4 - \rho_1\rho_4 &= 0, \\ \lambda_2\lambda_4 + \mu_2\mu_4 + \nu_2\nu_4 - \rho_2\rho_4 &= 0, \\ \lambda_3\lambda_4 + \mu_3\mu_4 + \nu_3\nu_4 - \rho_3\rho_4 &= 0, \end{aligned}$$

and x, y, z, w be the coordinates of a circle C referred to D_1, D_2, D_3, D_4, then putting

$$C = xD_1 + yD_2 + zD_3 + wD_4 = xC_1 + yC_2 + zC_3 + wC_4,$$

and substituting, we have the following expressions for the old coordinates in terms of the new

$$\begin{aligned} x &= \lambda_1 X + \lambda_2 Y + \lambda_3 Z + \lambda_4 W, \\ y &= \mu_1 X + \mu_2 Y + \mu_3 Z + \mu_4 W, \\ z &= \nu_1 X + \nu_2 Y + \nu_3 Z + \nu_4 W, \\ -w &= \rho_1 X + \rho_2 Y + \rho_3 Z + \rho_4 W. \end{aligned}$$

Also
$$X = D_1 C = \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 w,$$
$$Y = D_2 C = \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 w,$$
$$Z = D_3 C = \lambda_3 x + \mu_3 y + \nu_3 z + \rho_3 w,$$
$$-W = D_4 C \quad \lambda_4 x + \mu_4 y + \nu_4 z + \rho_4 w.$$

Again, since
$$C_1 = (C_1 D_1) D_1 + (C_1 D_2) D_2 + (C_1 D_3) D_3 - (C_1 D_4) D_4,$$
we have
$$C_1 = \lambda_1 D_1 + \lambda_2 D_2 + \lambda_3 D_3 - \lambda_4 D_4,$$
since by
$$C_2 = \mu_1 D_1 + \mu_2 D_2 + \mu_3 D_3 - \mu_4 D_4,$$
$$C_3 = \nu_1 D_1 + \nu_2 D_2 + \nu_3 D_3 - \nu_4 D_4,$$
$$C_4 = \rho_1 D_1 + \rho_2 D_2 + \rho_3 D_3 - \rho_4 D_4.$$

Hence, since
$$C_1^2 = 1,\ C_2^2 = 1,\ C_3^2 = 1,\ C_4^2 = -1,\ C_1 C_2 = 0 \ldots,$$
we have
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_4^2 = 1,$$
$$\mu_1^2 + \mu_2^2 + \mu_3^2 - \mu_4^2 = 1,$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 - \nu_4^2 = 1,$$
$$\rho_1^2 + \rho_2^2 + \rho_3^2 - \rho_4^2 = -1,$$
$$\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 - \lambda_4\mu_4 = 0,$$
$$\lambda_1\nu_1 + \lambda_2\nu_2 + \lambda_3\nu_3 - \lambda_4\nu_4 = 0,$$
$$\lambda_1\rho_1 + \lambda_2\rho_2 + \lambda_3\rho_3 - \lambda_4\rho_4 = 0,$$
$$\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3 - \mu_4\nu_4 = 0,$$
$$\mu_1\rho_1 + \mu_2\rho_2 + \mu_3\rho_3 - \mu_4\rho_4 = 0,$$
$$\nu_1\rho_1 + \nu_2\rho_2 + \nu_3\rho_3 - \nu_4\rho_4 = 0.$$

These formulæ are identical with those I have given in a paper on "Homogeneous Coordinates in Imaginary Geometry."*

Another system of coordinates is sometimes convenient. We may refer the circle C to two circles C_1, C_2 at right angles, and to their points of intersection P_1, P_2, so, that
$$C = x C_1 + y C_2 + z P_1 + w P_2.$$

* The analogy between the geometry of systems of circles and that of non-euclidear space of three dimensions, is pointed out by Mehmke.

We will choose the multiplicity of C_1, C_2, P_1, P_2, so that if r_1, r_2 be the radii of C_1, C_2, and a the distance between P_1 and P_2,

$$\Theta C_1 = \frac{1}{r_1},\ \Theta C_2 = \frac{1}{r_2},\ \Theta P_1 = \frac{1}{a},\ \Theta P_2 = \frac{1}{a},$$

and therefore

$$C_1^2 = 1,\ C_2^2 = 1,\ C_1 C_2 = 0,\ P_1^2 = 0,\ P_2^2 = 0,\ P_1 P_2 = -\tfrac{1}{2}.$$

The condition that C should be a point is

$$x^2 + y^2 - zw = 0.$$

In this case, if

$$P = xC_1 + yC_2 + zP_1 + wP_2,$$

and t_1, t_2 be the tangents from P to C_1, C_2, d_1, d_2, its distances from P_1, P_2,

$$x = PC_1 = -\frac{t_1^2}{2r_1},\ y = PC_2 = -\frac{t_2^2}{2r_2},$$

$$z = \frac{PP_1^2}{P_1P_2} = \frac{d_2^2}{a},\ w = \frac{d_1^2}{a},$$

and substituting

$$\frac{t_1^4}{r_1^2} + \frac{t_2^4}{r_2^2} = 4\frac{d_1^2 d_2^2}{a^2}.$$

In particular

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{4}{a^2},$$

a relation easily verified. If

$$P = xC_1 + yC_2 + zP_1 + wP_2,$$

be a point or multiple of a point,

$$P' = -xC_1 + yC_2 + zP_1 + wP_2,$$

is also a multiple of a point, and since $P - P' = 2C_1$ the circle C_1 may be derived from them. Hence if O_1 be the centre of their circles,

$$\text{dist. } O_1P \times \text{dist. } O_1P' = r_1^2.$$

Now if d_1', d_2' be the distances of P' from P_1, P_2, then since

$$-\tfrac{1}{2}d_1^2 = \frac{PP_1}{\Theta P.\Theta P_1} = -\frac{\frac{1}{2}w}{\Theta P.\Theta P_1},\ -\tfrac{1}{2}d_2^2 = -\frac{\frac{1}{2}z}{\Theta P.\Theta P_1},$$

we have $$\frac{d_1^2}{d_2^2} = \frac{w}{z} = \frac{d_1'^2}{d_2'^2},$$

or $$\frac{d_1}{d_1'} = \frac{d_2}{d_2'}.$$

Since C_2 may be chosen so that P_2 is any other point on the circle C_1, this gives the theorem, that the curve of a point, the ratio of whose distances from two fixed points is constant, is a circle.

Bicircular Quartics. Reduction of the general Equation of the Second Degree.

The general equation of the second degree may be written

$$ax^2 + by^2 + cz^2 + dw^2 + 2lyz + 2mzx + 2nxy + 2pxw + 2qyw + 2rzw = 0.$$

If we consider only points, we must add the equation

$$x^2 + y^2 + z^2 - w^2 = 0.$$

Or, substituting in Cartesian coordinates, it is easy to see that the curves represented by this equation are bicircular quartics. They may be defined as curves which are cut by a circle in four points, two or four of which may however be imaginary.

To simplify this equation, we will introduce new coordinates X, Y, Z, W, and make the coefficients XY, XZ, XW, YZ, ZW, YW disappear.

For the disappearanee of XY, XZ, XW, we have the equations

$$\lambda_2(a\lambda_1 + n\mu_1 + m\nu_1 - p\rho_1) + \mu_2(n\lambda_1 + b\mu_1 + l\nu_1 - q\rho_1) + \nu_2(m\lambda_1 + l\mu_1 + c\nu_1 - r\rho_1) - \rho_2(p\lambda_1 + q\mu_1 + r\nu_1 - d\rho_1) = 0,$$

$$\lambda_3(a\lambda_1 + n\mu_1 + m\mu_1 - p\rho_1) + \mu_3(n\lambda_1 + b\mu_1 + l\nu_1 - p\rho_1) + \nu_3(m\lambda_1 + l\mu_1 + c\nu_1 - r\rho_1) - \rho_3(p\lambda_1 + q\mu_1 + r\nu_1 - d\rho_1) = 0,$$

$$\lambda_4(a\lambda_1 + n\mu_1 + m\mu_1 - p\rho_1) + \mu_4(n\lambda_1 + b\mu_1 + l\nu_1 - p\rho_1) + \nu_4(m\lambda_1 + l\mu_1 + c\nu_1 - r\rho_1) - \rho_4(p\lambda_1 + q\mu_1 + r\nu_1 - d\rho_1) = 0.$$

Combining these with the equations

$$\lambda_2\lambda_1 + \mu_2\mu_1 + \nu_2\nu_1 - \rho_2\rho_1 = 0,$$

$$\lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1 - \rho_3\rho_1 = 0,$$

$$\lambda_4\lambda_1 + \mu_4\mu_1 + \nu_4\nu_1 - \rho_4\rho_1 = 0,$$

we have

$$\frac{a\lambda_1+n\mu_1+m\nu_1-p\rho_1}{\lambda_1}=\frac{n\lambda_1+b\mu_1+l\nu_1-q\rho_1}{\mu_1}=\frac{m\lambda_1+l\mu_1+c\nu_1-r\rho_1}{\nu_1}$$

$$=\frac{p\lambda_1+q\mu_1+r\nu_1-d\rho_1}{\rho_1}=h \text{ say.}$$

Eliminating, we get the equations

$$\begin{vmatrix} a-h, & n, & m, & p \\ n, & b-h, & l, & q \\ m, & l, & c-h, & r \\ p, & q, & r, & d+h \end{vmatrix}=0.$$

The equations for

$$(\lambda_2\mu_2\nu_2\rho_2),\ (\lambda_3\mu_3\nu_3\rho_3),\ (\lambda_4\mu_4\nu_4\rho_4),$$

are precisely similar, so that if A, B, C, D be the roots of the above biquadratic equation, the values $\lambda_1\mu_1\nu_1\rho_1$, $\lambda_2\mu_2\nu_2 d_2$, &c. are obtained by substituting successively A, B, C, D for h. Conversely, it is easy to shew that where A, B, C, D are all different, the new circles so determined form an orthogonal system.

Since $$\lambda_1^2+\mu_1^2+\nu_1^2-\rho_1^2=1,$$

we have

$$a\lambda_1^2+b\mu_1^2+c\nu_1^2+d\rho_1^2+2l\mu_1\nu_1+2m\nu_1\lambda_1+2n\lambda_1\mu_1$$
$$-2p\lambda_1\rho_1-2q\mu_1\rho_1-2r\nu_1\rho_1=A,$$

and since $$\lambda_4^2+\mu_4^2+\nu_4^2-\rho_4^2=-1,$$

$$a\lambda_4^2+b\mu_4^2+c\nu_4^2+d\rho_4^2+2l\mu_4\nu_4+2m\nu_4\lambda_4+2n\lambda_4\mu_4$$
$$-2p\lambda_4\rho_4-2q\mu_4\rho_4-2r\nu_4\rho_4=-D.$$

Hence the new equation will be

$$AX^2+BY^2+CZ^2=DW^2.$$

When two of the roots of the biquadratic are equal, we can still choose either $\lambda_1\mu_1\nu_1\rho_1$, $\lambda_2\mu_2\nu_2\rho_2$, or $\lambda_3\mu_3\nu_3\rho_3$, $\lambda_4\mu_4\nu_4\rho_4$ to correspond to the remaining two, so that the equation will reduce to either

$$ax^2+by^2+cz^2+dw^2+2nxy=0,$$

or to $$ax^2+by^2+cz^2+dw^2+2rzw=0.$$

In the first case the biquadratic becomes

$$(c-h)(d+h)\{h^2-(a+b)h+ab-n^2\}=0,$$

and it will have equal roots if

$$(a+b)^2=4(ab-n^2),$$

or

$$(a-b)^2+4n^2=0,$$

that is

$$a=b,\ n=0.$$

The equation then becomes

$$a(x^2+y^2)+cz^2+dw^2=0,$$

or since

$$x^2+y^2+z^2-w^2=0,$$

$$(c-a)z^2+(d+a)w^2=0.$$

It therefore represents two circles and need not be considered.

In the other case the biquadratic becomes

$$(a-h)(b-h)\{cd-r^2+h(c-d)-h^2\}=0.$$

It has equal roots if

$$(c-d)^2=4(r^2-cd),$$

or

$$c+d=\pm 2r.$$

If then from the equation

$$ax^2+by^2+cz^2+dw^2\pm(c+d)wz=0,$$

we subtract $\frac{1}{2}(c-d)(x^2+y^2+z^2-w^2)$,
we get

$$(a-\tfrac{1}{2}c+\tfrac{1}{2}d)x^2+(b-\tfrac{1}{2}c+\tfrac{1}{2}d)y^2+\tfrac{1}{2}(c+d)(z^2+w^2\pm 2wz)=0.$$

Now $z\pm w=P(C_3\pm C_4)$ and $(C_3\pm C_4)^2=C_3^2+C_4^2=0$.

Hence $C_3\pm C_4$ is one of the points of intersection P_1P_2 of the circles C_1C_2.

If P be referred to C_1C_2, P_1P_2, so that

$$P=XC_1+YC_2+ZP_1+WP_2,$$

then $z\pm w=XY$ some multiple of PP_1 or PP_2 = some multiples of W or Z, and the equation may be written

$$AX^2+BY^2=CZ^2,$$

$$X^2+Y^2=WZ.$$

Finally, then every bicircular quartic may be represented by one of the following pairs of equations,

$$\left.\begin{aligned} ax^2 + by^2 + cz^2 &= dw^2 \\ x^2 + y^2 + z^2 &= w^2 \end{aligned}\right\} \text{ ..... I,}$$

$$\left.\begin{aligned} ax^2 + by^2 &= cz^2 \\ x^2 + y^2 &= wz \end{aligned}\right\} \text{ .....II.}$$

Some Properties of Bicircular Quartics.

We take the first set of equations. It is clear that if x, y, z, w be a point on the curve, $-x, y, z, w$ is also a point on the curve.

Let P, P' be these points. Then

$$P = xC_1 + yC_2 + zC_3 + wC_4,$$

$$P' = -xC_1 + yC_2 + zC_3 + wC_4.$$

Therefore $$P - P' = Qx C_1.$$

Hence $$\text{dist. } O_1P \times \text{dist. } O_1P' = r_1^2.$$

When P and P' coincide, then $O_1P = r_1$, so that P is on the circle C_1. But then O_1P will be a tangent to the quartic. Thus each of the four circles cuts the quartic at right angles in four points.

Again, if O_1QQ' be another line through O_1,

$$O_1Q \times O_1Q' = r_1^2.$$

Therefore a circle can be drawn through P, P', Q, Q', and it will cut the circle C_1 at right angles. When Q coincides with P and Q' with P', we have a circle having double contact with the curve; and when P, P', Q, Q' all coincide, we have a circle having four-point contact with the curve, so that the points where the fundamental circles meet the curve are points where a circle can have four-point contact. In the equations I, we may take d positive and $a > b > c$. Then we must have $a > d$, $c < d$. The points where C_1 meets the curve are given by

$$by^2 + cz^2 = dw^2,$$

$$y^2 + z^2 = w^2,$$

or $$\frac{y^2}{d-c} = \frac{z^2}{b-d} = \frac{w^2}{b-c}.$$

They will be real or imaginary according as $b >$ or $< d$.

The points where C_2 meets the curve are given by

$$\frac{x^2}{d-c} = \frac{z^2}{a-d} = \frac{w^2}{a-c}.$$

They are always real.

The points where C_3 meets the curve are given by

$$\frac{x^2}{a-d} = \frac{y^2}{d-b} = \frac{w^2}{a-b}.$$

They are real or imaginary according as $d >$ or $< b$. C_4 has no real points.

Thus the curve meets c_1c_2 or c_2c_3 according as $b >$ or $< d$. The four points on one of the circles form a quadrilateral having the centres of the other circles for summits. There are then eight real points where a circle has four-point contact.

Any circle l, m, n, r touching the curve at x, y, z, w must satisfy the equations

$$lx + my + nz - rw = 0,$$

$$l\delta x + m\delta y + n\delta z - r\delta w = 0;$$

but

$$ax\delta x + by\delta y + cz\delta z - dw\delta w = 0,$$

$$x\delta x + y\delta y + z\delta z - w\delta w = 0.$$

Hence we may put

$$l = ax + \lambda x,\ m = by + \lambda y,\ m = cz + \lambda z,\ r = dw + \lambda w.$$

When $\lambda = -a$, we have a circle whose coordinates are 0, $(a-b)y$, $(a-c)z$, $(a-d)w$ which will touch the curve, not only in x, y, z, w, but also in $-x, y, z, w$. The coordinates $0, m, n, r$ of this circle satisfy the equation

$$\frac{m^2}{a-b} + \frac{n^2}{a-c} = \frac{r^2}{a-d}.$$

If it reduces to a point

$$m^2 + n^2 = r^2.$$

In this case the point is called a focus, so that there are four foci on each of the four circles. We have

$$\frac{m^2}{(a-b)(d-c)} = \frac{n^2}{(a-c)(b-d)} = \frac{r^2}{(a-d)(b-c)}.$$

These points are real if $b > d$.

The foci on C_2 are given by

$$\frac{l^2}{(b-a)(d-c)} = \frac{n^2}{(b-c)(a-d)} = \frac{r^2}{(b-d)(a-c)}.$$

They are always imaginary.

Those on C_3 are given by

$$\frac{l^2}{(c-a)(a-d)} = \frac{m^2}{(c-b)(d-b)} = \frac{w^2}{(c-d)(a-b)}.$$

They are read if $d > b$.

Thus there are four real foci lying on that one of the circles $C_1 C_3$ which is met by the curve in real points.

The condition that the circle $mC_2 + nC_3 + rC_4$ should be a straight line is

$$\frac{m}{r_2} + \frac{n}{r_3} + \frac{r}{r_4} = 0;$$

combining this with the equation

$$\frac{m^2}{a-b} + \frac{n^2}{a-c} = \frac{r^2}{a-d},$$

which expresses that the circle has double contact, we shall obtain two double tangents from the centres of each of the four fundamental circles.

The ratio $m : n$ is given by

$$\left(\frac{m^2}{a-b} + \frac{n^2}{a-c}\right)\frac{1}{r_4^2} = \left(\frac{m}{r_2} + \frac{n}{r_3}\right)^2 \frac{1}{a-d}.$$

Its roots are real or imaginary according as

$$\left\{\frac{1}{(a-d)r_2^2} - \frac{1}{(a-b)r_4^2}\right\}\left\{\frac{1}{(a-d)r_3^2} - \frac{1}{(a-c)r_4^2}\right\},$$

$$< \text{ or } > \frac{1}{(a-d)^2 r_2^2 r_3^2};$$

that is as

$$\frac{-1}{(a-b)(a-c)(a-d)}\left(\frac{a-b}{r_2^2} + \frac{a-c}{r_3^2} - \frac{a-d}{r_4^2}\right) < \text{ or } > 0,$$

or as

$$\frac{1}{(a-b)(a-c)(a-d)}\left(\frac{a}{r_1^2} + \frac{b}{r_2^2} + \frac{c}{r_3^2} - \frac{d}{r_4^2}\right) < \text{ or } > 0.$$

For the other three circles the conditions are

$$\frac{1}{(b-a)(b-c)(b-d)}\left(\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}-\frac{d}{r_4^2}\right)<\text{or}>0,$$

$$\frac{1}{(c-a)(c-b)(c-d)}\left(\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}-\frac{d}{r_4^2}\right)<\text{or}>0,$$

$$\frac{1}{(a-d)(b-d)(c-d)}\left(\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}-\frac{d}{r_4^2}\right)<\text{or}>0.$$

When then

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}-\frac{d}{r_4^2}<0,$$

the double tangents from O_1 are real, those from O_3 imaginary; those from O_2, O_4 are imaginary if $b>d$, real if $b<d$. The opposite holds if

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}-\frac{d}{r_4^2}>0.$$

Hence, either the two double tangents from the centre of the circle which has the four real foci are real, or else the six double tangents form the centres of the other circles are real. Since the curve (supposing $b>d$) does not cut C_3, and to every point P corresponds a point P', such that $O_3P.O_3P'=r_3^2$, it must consist of two ovals, one entirely inside C_3 and one entirely outside. A tangent from O_3 to of one the ovals must touch the other. Hence, in the first case O_3 must lie inside both ovals, and so also must O_4. One oval will then entirely inclose the other. In the second case neither of the ovals contains O_3 or O_4, and they will lie entirely outside one another.

If P, P' be the points of contact of one double tangent from O_1, QQ' those of the other, then

$$O_1P.O_1P'=O_1Q.O_1Q'=r_1^2,$$

so that PP', QQ' lie on a circle cutting C_1 at right angles. Thus the sixteen points of contact of the eight double tangents lie four by four on four circles.

To find the circle of curvature at a point x, y, z, w on the curve, we have to make the circle

$$(a^2+\lambda)x,\ (b^2+\lambda)y,\ (c^2+\lambda)z,\ (d^2+\lambda)w$$

pass through a third consecutive point, or to make

$$(a+\lambda)\,xd^2x+(b+\lambda)\,yd^2y+(c+\lambda)\,zd^2z=(d+\lambda)\,wd^2w.$$

The problem is the same as that of finding the osculating plane to a curve given as the intersection of two quadrics, and its solution is given in Salmon's *Solid Geometry*, (p. 290 and 291, Ex. 1, 2nd ed).

Since only the ratio of x, y, z, w are important, we may take any one of these quantities, x for instance, to be constant.

Then $$(b+\lambda)\,yd^2y+(c+\lambda)\,zd^2z=(d+\lambda)\,wd^2w,$$

and $$bydy+cydz=dw\,dw,$$

$$ydy+zdz=wdw.$$

Differentiating again, and substituting

$$(b+\lambda)\,(dy)^2+(c+\lambda)\,(dz)^2=(d+\lambda)\,(dw)^2.$$

But $$\frac{dy}{-(c-d)\,zw}=\frac{dz}{(b-d)\,yw}=\frac{dw}{(b-c)\,xy}.$$

So that

$$(b+\lambda)\,(c-d)^2w^2z^2+(c+\lambda)\,(b-d)^2w^2y^2=(d+\lambda)\,(b-c)^2z^2y^2.$$

Hence

$$b+\lambda=\frac{y^2\,(b-c)\,(b-d)\,\{(b-d)\,w^2-(b-c)\,z^2\}}{(c-d)\,w^2z^2+(b-d)^2w^2y^2-(b-c)^2z^2y^2}.$$

We may take

$$b+\lambda \text{ prop. to } y^2\,(b-a)\,(b-c)\,(b-d),$$

since $$(b-d)\,w^2-(b-c)\,z^2=(b-a)\,x^2.$$

Similarly we may take

$$(c+\lambda) \text{ prop. to } \frac{z^2}{x^2}\,(c-b)\,(c-d)\,\{(c-d)\,w^2-(c-b)\,y^2\},$$

$$\text{to} \quad (c-a)\,(c-b)\,(c-d)\,z^2,$$

and $$d+\lambda \text{ to } \frac{w^2}{x^2}\,(d-b)\,(d-c)\,\{(d-b)\,y^2+(d-c)\,z^2\}$$

$$=w^2\,(a-d)\,(b-d)\,(c-d).$$

By putting any one of the quantities x, y, z, w equal to 0, we get

$$a+\lambda \text{ prop. to } x^2\,(a-b)\,(a-c)\,(a-d).$$

Hence the coordinates of the circle of curvature are

$$(a-b)(a-c)(a-d)x^3, \quad (b-a)(b-c)(b-d)y^3,$$
$$(c-a)(c-b)(c-d)z^3, \quad (a-d)(b-d)(c-d)w^3.$$

If this be a straight line

$$\frac{(a-b)(a-c)(a-d)x^3}{r_1} + \frac{(b-a)(b-c)(b-d)y^3}{r_2}$$
$$+ \frac{(c-a)(c-b)(c-d)z^3}{r_3} + \frac{(a-d)(b-d)(c-d)w^3}{r_4} = 0.$$

In this case (x, y, z, w) will be a point of inflexion, and combining this equation with the other two it follows that there are twelve points of inflexion. The curve that determines the points of inflexion by its points of intersection with the original curve is a tricircular cubic.

If I, I' be the circles of curvature at two corresponding points P, P', such that $O_1P.\,O_1P' = r_1^2$, then we shall have

$$I - I' = 2(a-b)(a-c)(a-d)x^3\,C_1.$$

Hence the centres of curvature F, F' lie on a straight line passing through O_1. Since a circle can be drawn having double contact with the curve in P, P', the normals $PF, P'F'$ make equal angles with PP'. Hence

$$\frac{PF}{O_1F} = \frac{\sin POF}{\sin OPF} = \frac{P'F'}{O_1F'}.$$

That is to say, the radii of curvature at two corresponding points are proportional to the distance of the centres of curvature from the centre of the fundamental circle; or the centre of the fundamental circle is the centre of similitude of the two circles of curvature.

The evolute of the bicircular quartic has sixteen cusps corresponding to the points on the curve where a circle can have four-point contact. The cusps corresponding to points on C_1 form a quadrilateral with O_2, O_3, O_4 for summit. Through each point O_1 can be drawn six lines containing twelve out of the sixteen cusps. The maximum radii of curvature for points on C_1 are such that the sums of their reciprocals, taken with proper signs, vanishes.

As the foci on C_1 are given by the two equations

$$\frac{m^2}{a-b} + \frac{n^2}{a-c} = \frac{r^2}{a-d}, \quad m^2 + n^2 = r^2,$$

they will be the same for any curve for which the differences of $\frac{1}{a-b}, \frac{1}{a-c}, \frac{1}{a-d}$ are the same. That is, if we put

$$a-b=\frac{1}{\rho^2+\lambda}, \quad a-c=\frac{1}{\gamma^2+\lambda}, \quad a-d=\frac{1}{\delta^2+\lambda},$$

where β, γ, δ are kept constant, all the curves obtained by varying λ will be confocal. If one of the series pass through x, y, z, w,

$$\frac{y^2}{\beta^2+\lambda}+\frac{z^2}{\gamma^2+\lambda}=\frac{w^2}{\delta^2+\lambda}.$$

This is a quadratic equation in λ; let λ_1, λ_2 be its roots; then

$$\frac{y^2}{\beta^2+\lambda_1}+\frac{z^2}{\gamma^2+\lambda_1}=\frac{w^2}{\delta^2+\lambda_1},$$

$$\frac{y^2}{\beta^2+\lambda_2}+\frac{z^2}{\gamma^2+\lambda_2}=\frac{w^2}{\delta^2+\lambda_2}.$$

Substituting

$$\frac{y^2}{(\beta^2+\lambda_1)(\beta^2+\lambda_2)}+\frac{z^2}{(\gamma^2+\lambda_1)(\gamma^2+\lambda_2)}=\frac{w^2}{(\delta^2+\lambda_1)(\delta^2+\lambda_2)}.$$

This expresses that the circles

$$0, \quad \frac{y}{\beta^2+\lambda_1}, \quad \frac{z}{\gamma^2+\lambda_1}, \quad \frac{w}{\delta^2+\lambda_1},$$

$$0, \quad \frac{y}{\beta^2+\lambda_2}, \quad \frac{z}{\gamma^2+\lambda_2}, \quad \frac{w}{\delta^2+\lambda_2},$$

cut at right angles. But these circles touch the two quartics respectively at x, y, z, w. Therefore the curves themselves cut at right angles, and we have the theorem that the two confocal bicircular quartics that can be drawn through any point cut at right angles.

If we put $\lambda=-\beta^2$ we must have $y=0$, so that the circle C_2 may be said to belong to the confocal system. But more specially (Salmon's *Solid Geometry*, p. 114) the points given by

$$\frac{z^2}{\gamma^2-\beta^2}=\frac{w^2}{\delta^2-\beta^2},$$

belong to the system.

On substituting, these are seen to be the points already found for the foci on C_2. Thus the sixteen foci belong in a special sense to the system of confocal quartics. If the four fundamental circles and one of the foci be given, all the others will be known. This is obvious from their coordinates. Geometrically, if F_1 be one of the foci on C_1, by joining O_2, O_3, O_4 to F_1, and producing the lines to meet the circle again, we get the other foci on C_1, F_2, F_3, F_4.

Now looking at the values of the coordinates of the foci on C_2, it is seen that two of them are imaginary circles at right angles to F_1, F_2, and the other two imaginary circles at right angles to F_3, F_4.

Or, in other words, two of the foci on C_2 are the imaginary points of the system of circles passing through F_1, F_2, and the other two are the imaginary points of circles passing through F_3, F_4. Similarly for the foci on C_3, C_4. The conjugate pairs of imaginary foci lie on six real lines. These lines pass through O, and are at right angles to the lines joining the real foci. When

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}-\frac{d}{r_4^2}=0,$$

the curve reduces to a cubic with a single infinite branch. For

$$\frac{1}{r_1},\ \frac{1}{r_2},\ \frac{1}{r_3},\ \frac{1}{r_4},$$

are the coordinates of the point Θ. This cubic must pass through O_1, O_2, O_3, O_4, and the tangents at these points will be parallel to the asymptote.

In the second kind of bicircular quartics given by the equations

$$ax^2+by^2=cz^2,$$

$$x^2+\ y^2=wz,$$

where $$P=xC_1+yC_2+zP_1+wP_2,$$

we may take a and c positive $a>b>c$.

P_2 is a double point. If b be positive, it is a conjugate point, and the rest of curve consist of an oval. If b be negative, it is the intersection of two branches. There are two real foci lying on C_1, and the line joining them passes through O_2. It is clear that for any point P we can find another on O_1, P, so that $O_1P.\ O_1P'=r_1^2$, and if P be on the curve P' will. The conic sections are included among these curves, and are obtained by making two of the circles straight lines.

NOTE ON A DIFFERENTIAL EQUATION DUE TO KUMMER.

By Prof. A. R. Forsyth.

IF y and z are two solutions of the equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + ry = 0,$$

and $s = \frac{y}{z}$, then writing

$$\{s, x\} \equiv \frac{s'''}{s'} - \tfrac{3}{2}\left(\frac{s''}{s'}\right)^2,$$

it is proved (cf. Cayley, *Camb. Phil. Trans.* vol. XIII.), that

$$\{s, x\} = -\tfrac{1}{2}\left(p^2 + 2\frac{dp}{dx} - 4r\right).$$

In particular we have K', K as solutions of

$$\frac{d^2K}{dc^2} + \left(\frac{1}{c} - \frac{1}{1-c}\right)\frac{dK}{dc} - \tfrac{1}{4}K\left(\frac{1}{c} + \frac{1}{1-c}\right) = 0,$$

where $c = k^2$; and therefore if $s = \frac{K'}{K}$, we have

$$\{s, c\} = -\tfrac{1}{2}\left[\left(\frac{1}{c} - \frac{1}{1-c}\right)^2 + 2\left(\frac{-1}{c^2} - \frac{1}{(1-c)^2}\right) + \frac{1}{c} + \frac{1}{1-c}\right]$$

$$= \frac{\frac{1}{2}}{c^2} + \frac{\frac{1}{2}}{(1-c)^2} + \frac{\frac{1}{2}}{c.1-c}$$

$$= \frac{1}{2c^2} + \frac{1}{2c} + 1 + \ldots + (\tfrac{1}{2}n + 1)\,c^n + \ldots.$$

Now write $$s = -\frac{1}{\pi}\log q.$$

The coefficient of $\log q$ will disappear from the result, and thus we shall expect that the part of the equation involving q will be covariantive for such a substitution as $q = v^n$.

We have

$$s' = -\frac{1}{\pi}\frac{q'}{q},$$

$$\frac{s''}{s'} = \frac{q''}{q'} - \frac{q'}{q},$$

$$\frac{s''}{s'} - \left(\frac{s''}{s'}\right)^2 = \frac{q'''}{q'} - \left(\frac{q''}{q'}\right)^2 - \frac{q''}{q} + \left(\frac{q'}{q}\right)^2;$$

therefore

$$\{s, c\} = \{q, c\} + \tfrac{1}{2}\left(\frac{q'}{q}\right)^2,$$

and the differential equation for q is

$$\{q, c\} + \tfrac{1}{2}\left(\frac{q'}{q}\right)^2 = \frac{\frac{1}{2}}{c^2} + \frac{\frac{1}{2}}{(1-c)^2} + \frac{\frac{1}{2}}{c(1-c)}.$$

Returning now to the linear differential equation above, in which p, r denote functions of x, write, after Kummer (*Crelle*, t. XV), $y = wv$, where v satisfies the equation

$$\frac{d^2v}{dz^2} + P\frac{dv}{dz} + Rv = 0,$$

P, Q being functions of z, z a function of x, and w a function of x only. Then it is shewn that

$$w^2\frac{dz}{dx} = Ce^{\int Pdz - \int pdx},$$

and, writing

$$P^2 + 2\frac{dP}{dz} - 4R = Z, \text{ a function of } z,$$

$$p^2 + 2\frac{dp}{dx} - 4r = X, \ldots\ldots\ldots\ldots x,$$

that

$$\{z, x\} - \tfrac{1}{2}Zz'^2 + \tfrac{1}{2}X = 0.$$

To make this agree with the particular example considered, let

$$Z = -\frac{1}{z^2},$$

or

$$P^2 + 2\frac{dP}{dz} - 4R = -\frac{1}{z^2}.$$

Take $P=\frac{\lambda}{z}$ and $R=\frac{\mu^2}{z^2}$; then

$$\lambda^2-2\lambda-4\mu^2=-1,$$

or
$$\lambda=1\pm 2\mu.$$

Hence the equation for v is

$$\frac{d^2v}{dz^2}+\frac{1\pm 2\mu}{z}\frac{dv}{dz}+\frac{\mu^2}{z^2}v=0,$$

and therefore
$$v=z\mp\mu\,(A'+B'\log z).$$

Substituting for P and p, we have

$$w^2\frac{dz}{dx}=C\frac{z^{1\pm 2\mu}}{x\,(1-x)}.$$

Now $w=\frac{y}{v}$; taking the above value of v, we have

$$\frac{dz}{dx}\cdot\frac{y^2}{(A'+B'\log z)^2}=C\frac{z}{x.1-x},$$

or, for particular values of the constants, either

$$\frac{y^2}{z}\frac{dz}{dx}=\frac{C}{x.1-x}$$

or
$$\frac{y^2}{(\log z)^2}\frac{d\log z}{dx}=\frac{C}{x.1-x}.$$

The first of these gives us, on taking $y=K$,

$$\frac{1}{q}\frac{dq}{dc}=\frac{C}{c.1-c.K^2}.$$

We need to determine C. Let c be very small; then K is approximately equal to $\frac{1}{2}\pi$, and

$$q=\tfrac{1}{16}c,$$

therefore
$$\tfrac{1}{4}\pi^2.\frac{1}{c}=\frac{C}{c},$$

and thus
$$\frac{1}{q}\frac{dq}{dc}=\frac{\pi^2}{4K^2}\frac{1}{c\,(1-c)},$$

a known formula.

This formula is perhaps the best suited to obtaining a series for q when c is small. We have

$$K = \tfrac{1}{2}\pi \left[1 + (\tfrac{1}{2})^2 c + \left(\frac{1.3}{2.4}\right)^2 c^2 + \ldots\right]$$

$$= \tfrac{1}{2}\pi \left[1 + b_1 c + b_2 c^2 + \ldots\right] \text{ say.}$$

Write $$\frac{\pi^2}{4K^2} = 1 + a_1 c + a_2 c^2 + \ldots ,$$

so that

$$a_1 = -2b_1,$$

$$a_2 = -2b_2 + 3b_1^2,$$

$$a_3 = -2b_3 + 6b_1 b_2 - 4b_1^3,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

then

$$\frac{1}{q}\frac{dq}{dc} = \frac{1}{c}\left[1 + c + c^2 + \ldots\right]\left[1 + a_1 c + a_2 c^2 + \ldots\right]$$

$$= \frac{1}{c}\left[1 + (1 + a_1) c^2 + (1 + a_1 + a_2) c^2 + \ldots\right],$$

and therefore

$$\log q = \text{const.} + \log c + (1 + a_1) c + \tfrac{1}{2}(1 + \alpha_1 + \alpha_2) c^2 + \ldots .$$

When c is very small, $q = \frac{1}{16}c$, and we have

$$\text{const.} = -\log 16$$

and therefore

$$\log q = \log \tfrac{1}{16} c + (1 + a_1) c + (1 + a_1 + a_2) c^2 + \ldots .$$

Let $$q = \frac{c}{16}\left[1 + A_1 c + A_2 c^2 + \ldots\right];$$

thus $(1 + a_1) c + (1 + a_1 + a_2) c^2 + \ldots = \log(1 + A_1 c^2 + A_2 c^2 + \ldots)$, and equating coefficients

$$A_1 = 1 + a_1,$$

$$A_2 - \tfrac{1}{2}A_1^2 = \tfrac{1}{2}(1 + a_1 + a_2),$$

$$A_3 - A_1 A_2 + \tfrac{1}{3}A_1^3 = \tfrac{1}{3}(1 + a_1 + a_2 + a_3),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

equations which determine $A_1, A_2, A_3, \ldots$.

Returning now to our general equations we saw that z is determined as a function of x by the equation

$$\{z, x\} - \tfrac{1}{2}Zz'^2 + \tfrac{1}{2}X = 0.$$

But if s be the quotient of two solutions of the first equation, then

$$\{s, x\} + \tfrac{1}{2}X = 0$$

and hence $$\{z, x\} - \tfrac{1}{2}Zz'^2 = \{s, x\}.$$

Now by a property of the Schwarzian derivative (cf. Cayley, l. c. § 2), we have

$$\{s, z\} = \left(\frac{dx}{dz}\right)^2 [\{s, x\} - \{z, x\}],$$

and hence $$\{s, z\} = -\tfrac{1}{2}Z,$$

i.e. $$\{s, z\} + \tfrac{1}{2}Z = 0,$$

showing that s is the quotient of two solutions of the equation

$$\frac{d^2v}{dr^2} + P\frac{dv}{dz} + Rv = 0.$$

This result can be obtained without the assumption of the foregoing property; for an obvious first integral of the equation is

$$\frac{z''}{z'} + Uz' = \frac{s''}{s'},$$

which leads to the equation

$$\{s, x\} = \{z, x\} + \tfrac{1}{2}z'^2\left[2\frac{dU}{dz} - U^2\right]$$

and gives for the determination of U

$$2\frac{dU}{dz} - U^2 = -Z,$$

a form analogous to Riccati's equation. Writing

$$U = -\frac{2}{t}\frac{dt}{dz},$$

this becomes $$\frac{d^2t}{dz^2} = t\tfrac{1}{4}Z,$$

and therefore $$t = ve^{-\frac{1}{2}\int P dz}.$$

Thus we have $$\frac{s''}{s'} = \frac{z''}{z'} - \frac{2}{t}\frac{dt}{dx};$$

therefore $$s' = \frac{C}{t^2}\frac{dz}{dx},$$

or $$s = A + C\int\frac{dz}{t^2}$$

$$= \frac{1}{t}\left[At + Ct\int\frac{dz}{t^2}\right].$$

But the expression within the bracket is known to be the second solution of

$$\frac{d^2t}{dz^2} = t \cdot \tfrac{1}{4}Z,$$

so that s is the quotient of two solutions of the t equation and therefore of the v equation.

Noticing the property in § 5 of Cayley's paper, we now have as the solution of the differential equation

$$\{z, x\} - \tfrac{1}{2}Zz'^2 + \tfrac{1}{2}X = 0,$$

the algebraical integral

$$\frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \delta y_2} = \frac{\alpha' w_1 + \beta' w_2}{\gamma' w_1 + \delta' w_2}$$

or $$Ay_1w_1 + By_1w_2 + Cy_2w_1 + Dy_2w_2 = 0,$$

y_1, y_2 being two solutions of

$$\frac{d^2y}{dx^2} = y \cdot \tfrac{1}{4}X,$$

and w_1, w_2 being two solutions of

$$\frac{d^2w}{dz^2} = w \cdot \tfrac{1}{4}Z.$$

The integral obviously contains three arbitrary constants, the number proper to an equation of the third order.

An easy example is obtained by assuming X and Z both to be constants.

As it is convenient to have the equations in the better known form, write

$$y = Ye^{\frac{1}{2}\int p dx}, \quad p^2 + 2\frac{dp}{dx} - X = 4r,$$

$$w = We^{\frac{1}{2}\int P dz}, \quad P^2 + 2\frac{dP}{dz} - Z = 4R,$$

and the equations become respectively

$$\frac{d^2Y}{dx^2} + p\frac{dY}{dx} + rY = 0,$$

$$\frac{d^2W}{dz^2} + P\frac{dW}{dz} + RW = 0,$$

and the integral is

$$AY_1W_1 + BY_1W_2 + CY_2W_1 + DY_2W_2 = 0.$$

The algebraically integrable cases when both of these equations represent hypergeometric series have been considered by Brioschi and Klein (in the *Mathematischen Annalen*), and in a memoir by Cayley in the *Camb. Phil. Trans.*, vol. XIII.

University College,
Liverpool.

ON SEMINVARIANTS.

By Prof. CAYLEY.

THE present paper is a somewhat fragmentary one, but it contains some results which seem to me to be worth putting on record.

I consider here not any binary quartic in particular, but the whole series $(a, b, c\chi x, y)^2$, $(a, b, c, d\chi x, y)^3$, &c.; or in a somewhat different point of view, I consider the indefinite series of coefficients $(a, b, c, d, e\ldots)$; here, instead of covariants and invariants, we have only seminvariants; viz., a seminvariant is a function reduced to zero by the operator

$$\Delta = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots;$$

for instance, seminvariants are

$$a,\ ac - b^2,\ a^2d - 3abc + 2b^3,\ a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$

$$ae - 4bd + 3c^2,\ ace - ad^2 - b^2e + 2bcd - c^3, \text{ &c.}$$

A seminvariant is of a certain degree θ in the coefficients, and of a certain weight w (viz. the coefficients $a, b, c, d, \ldots$ are reckoned as being of the weights 0, 1, 2, 3... respectively); it is, moreover, of a certain rank ρ; viz. according as the highest letter therein is $a, c, d, e, \ldots$ (it is never b) the rank is taken to be 0, 2, 3, 4, ..., and we have $w =$ or $< \frac{1}{2}\rho\theta$.

The seminvariant may be regarded as belonging to a quantic $(a, \ldots)(x, y)^n$, the order of which, n, is equal to or greater than ρ; viz. in regard to such quantic the seminvariant, say A, is the leading coefficient of a covariant

$$(A, B, \ldots K)(x, y)^\mu,$$

where the weights of the successive coefficients are w, $w+1$, ... up to $n\theta - w$; hence number of terms less unity, that is μ, is $= n\theta - 2w$; the least value of μ is thus $= \rho\theta - 2w$, which is either zero, or positive; in the former case, $w = \frac{1}{2}\rho\theta$, the seminvariant is an invariant of the quantic $(a, \ldots)(x, y)^\rho$, the order of which is equal to the rank of the seminvariant; but if $w < \frac{1}{2}\rho\theta$, then it is the leading coefficient of a covariant $(A, B \ldots K)(x, y)^{\rho\theta - 2w}$ of the same quantic $(a, \ldots)(x, y)^\rho$; and in every case, taking $n > \rho$, the seminvariant is the leading coefficient A of a covariant

$$(A, B, \ldots Q)(x, y)^{n\theta - 2w}$$

of a quantic $(a, \ldots)(x, y)^n$.

Take A as belonging to the quantic $(a, \ldots)(x, y)^n$; corresponding to such quantic we have an operator Λ of the same rank n, viz.

$$\begin{aligned}
\Lambda &= 2b\partial_a + c\partial_b && \text{for } n = 2,\\
&= 3b\partial_a + 2c\partial_b + d\partial_c && \text{,, } 3,\\
&= 4b\partial_a + 3c\partial_b + 2d\partial_c + e\partial_d && \text{,, } 4,\\
&\ \ \vdots && \ \ \vdots
\end{aligned}$$

Operating with Λ on A, we have a series of terms

$$A,\ \Lambda A,\ \Lambda^2 A, \ldots \Lambda^{n\theta - 2w} A,$$

but the next term $\Lambda^{n\theta - 2w + 1} A$, and of course every succeeding term, is $= 0$, and this being so, the coefficients of the covariant $(A, B, \ldots K)(x, y)^{n\theta - 2w}$

are $$(1, \tfrac{1}{1}\Lambda, \tfrac{1}{1.2}\Lambda^2, \ldots) A,$$

or what is the same thing, each coefficient is obtained from the next preceding one by the formulæ

$$B = \tfrac{1}{1}\Lambda A,\ C = \tfrac{1}{2}\Lambda B,\ D = \tfrac{1}{3}\Lambda C, \ldots .$$

The coefficients A and K, B and J, ... are derived one from the other by reversal of the order of the coefficients of $(a, b, \ldots)(x, y)^n$, with or without a change of sign, and thus it is only necessary to calculate up to the middle coefficient, or pair of coefficients, and we obtain, moreover, a verification.

Calculating in this manner the covariant

$$(A, B, \ldots K)^{\rho\theta-2w},$$

which belongs to the quantic $(a, \ldots)(x, y)^\rho$, if we herein change $a, b, c, \ldots$ into $ax+by$, $bx+cy$, $cx+dy$, ... we obtain the covariant belonging to the quantic $(a, \ldots)(x, y)^{\rho+1}$; and in this covariant making the like change, or what is the same thing, in the first-mentioned covariant changing $a, b, c, \ldots$ into $(a, b, c)(x, y)^2$, $(b, c, d)(x, y)^2$, $(c, d, e)(x, y)^2$, ... we have the covariant belonging to $(a, \ldots)(x, y)^{\rho+2}$; and in like manner we obtain the covariant belonging to the quantic $(a, \ldots)(x, y)^n$ of any given order n.

In particular, if $w = \frac{1}{2}\rho\theta$, that is, if the given seminvariant be an invariant of $(a, \ldots)(x, y)^\rho$, then we obtain the series of covariants directly from A by therein changing $a, b, c, \ldots$ into $ax+by$, $bx+cy$, $cx+dy$, ... and in the result making the like change; or what is the same thing, in A changing $a, b, c, \ldots$ into $(a, b, c)(x, y)^2$, $(b, c, d)(x, y)^2$ $(c, d, e)(x, y)^2$, ...: and so on until we obtain the covariant for the quantic $(a, \ldots)(x, y)^n$ of the given order n.

A seminvariant which cannot be expressed as a rational and integral function of lower seminvariants is said to be irreducible. The theory is distinct from that of the irreducible covariants of a quantic of a given order; for instance, as regards the cubic $(a, b, c, d)(x, y)^3$, we have the irreducible covariant (invariant)

$$a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$

but this is not an irreducible seminvariant; it is

$$= (ac - b^2)(ae - 4bd) + 3c^2)$$
$$- a \,.\, (ace - ad^2 - b^2e - c^3 + 2bcd),$$

(or, what is the same thing, there is not for the quartic $(a, b, c, d, e)(x, y)^4$, or for the higher quantics, any *irreducible* covariant having the leading coefficient $a^2d^2 \ldots - 3b^2c^2$).

We may consider the question to determine the number of asyzygetic seminvariants of a given degree and weight. For instance, taking the weights up to 12, so that the series

of letters extends as far as m, then for the degrees 1, 2, 3 we have as follows:

$W=$	0	1	2	3	4	5	6	7	8	9	10	11	12
Deg. 1	a	b	c	d	e	f	g	h	i	j	k	l	m
Nos.	1	1	1	1	1	1	1	1	1	1	1	1	1
Diff.	0	0	0	0	0	0	0	0	0	0	0	0	0
Deg. 2	a^2	ab	ac b^2	ad bc	ae bd c^2	af be cd	ag bf ce d^2	ah bg cf de	ai bh cg df e^2	aj bi ch dg ef	ak bj ci dh eg f^2	al bk cj di eh fg	am bl ck dj ei fh g^2
Nos.	1	1	2	2	3	3	4	4	5	5	6	6	7
Diff.	1	0	1	0	1	0	1	0	1	0	1	0	1
Deg. 3	a^3	a^2b	a^2c ab^2	a^2d abc b^3	a^2e abd ac^2 b^2c	a^2f abe acd b^2d bc^2	a^2g abf ace ad^2 b^2e bcd c^3	a^2h abg acf ade b^2f bce bd^2 c^2d	a^2i abh aeg adf ae^2 b^2g bcf bde c^2e cd^2	a^2j abi ach adg aef b^2h beg bdf be^2 c^2f cde d^3	a^2k abj aci adh aeg af^2 b^2i bch bdg bef c^2g cef ce^2 d^2e	a^2l abk acj adi aeh afg b^2j bci bdh beg bf^2 c^2h cdg caf d^2f de^2	a^2m abl ack adj aei afh ag^2 b^2k bcj bdi beh bfg c^2i cdh ceg cf^2 d^2g def e^3
Nos.	1	1	2	3	4	5	7	8	10	12	14	16	19
Diff.	1	0	1	1	1	2	1	2	2	2	2	2	3

For the degree 1, the line of differences shows that the only seminvariant is ($W=0$), the seminvariant a.

For the degree 2, the line of differences 1, 0, 1, 0 ..., shows that the number of seminvariants is $=1$ for each even degree, $=0$ for each odd degree; thus for the weight 0 there is a seminvariant $=a^2$, which of course is not irreducible; while for each of the other even weights we have a single irreducible seminvariant; as well known the forms are:

$W=$	2	4	6	8	10	12
	$ac+1$ b^2-1	$ae+1$ $bd-4$ c^2+3	$ag+1$ $bf-6$ $ce+15$ d^2-10	$ac+1$ $bh-8$ $cg+28$ $df-56$ e^2+35	$ak+1$ $bj-10$ $ci+45$ $dh-120$ $eg+210$ f^2-126	$am+1$ $bl-12$ $ck+66$ $dj-220$ $ei+495$ $fh-792$ g^2+462

For degree 3, the line of differences shows, that for

W	$=0$	1	2	3	4	5	6	7	8	9	10	11	12
Nos. are	$=1$	0	1	1	1	1	2	1	2	2	2	2	3 ;

but inasmuch as for each even weight there is a quadric seminvariant, which multiplied by a gives a cubic seminvariant, to obtain the number of irreducible cubic seminvariants, we subtract

1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	0	1	0	1	1	1	1	2	1	2	2 ,

or the numbers of irreducible cubic seminvariants are as in the line last written down.

There is a convenience however in giving for each even weight, as well the rejected reducible covariant; and the entire series of results is found to be

$W+$	0	2	3	4	5	6		7	8	
	a^2+1	a^2c+1	a^2d+1	a^2e+1	a^2f+1	a^2g+1		a^2h+1	a^2i+1	
		ab^2-1	$abc-3$	$abd-4$	$abc-5$	$abf-6$		$abg-7$	$abh-8$	
			b^3+2	ac^2+3	$acd+2$	$ace+15$	$+1$	$acf+9$	$acg+28$	$+1$
					b^2d+8	ad^2-10	-1	$ade-5$	$adf-56$	-3
					bc^2-6	b^2e	-1	b^2f+12	ae^2+35	$+2$
						bcd	$+2$	$bce-30$	b^2g	-1
						c^3	-1	bd^2+20	bcf	$+3$
								c^2d	bde	-1
									c^2e	-3
									cd^2	$+2$

9			10			11			12		
a^2j	$+1$		a^2k	$+1$		a^2l	$+1$		a^2m	$+1$	
abi	-9		abj	-10		abk	-11		abl	-12	
ach		$+2$	aci	$+45$	$+1$	acj	$+35$	$+2$	ack	$+66$	$+3$
adg	$+42$	-7	adh	-120	-4	adi	-75	-9	adj	-220	-15
aef	-36	$+5$	aeg	$+210$	$+8$	aeh	$+90$	$+14$	aei	$+495$	$+40+1$
b^2h	$+36$	-2	af^2	-126	-5	afg	-42	-7	afh	-792	$-70-4$
bcg	-126	$+7$	b^2i		-1	b^2j	$+20$	-2	ag^2	$+462$	$+42+3$
bdf	-108	$+22$	bch		$+4$	bci	-90	$+9$	b^2k		-3
be^2	$+180$	-25	bdg		-4	bdh	$+240$	$+16$	bcj		$+15$
c^2f	$+270$	-27	bef		$+2$	beg	-420	-63	bdi		$-25-4$
cde	-450	$+45$	c^2g		-4	bf^2	$+252$	$+42$	beh		$+30+12$
d^3	$+200$	-20	cdf		$+8$	c^2h		-30	bfg		$-14-8$
			ce^2		-5	cdg		$+70$	c^2i		$-15+3$
			d^2e			cef		-21	cdh		$+40-8$
						d^2f		-56	ceg		$-70-22$
						de^2		$+35$	cf^2		$+42+24$
									d^2g		$+24$
									def		-36
									e^3		$+15$

The canonical form given for the quintic in my Tenth Memoir on Quintics belongs to a series, viz. writing now the

the small roman letters (instead of the italic letters) for the series of coefficients, and using the italic letters $a, c, d, e, f \ldots$ to denote seminvariants as follows:

0 $a = \mathrm{a}$,

2 $c = \mathrm{ac} - \mathrm{b}^2$,

3 $d = \mathrm{a}^2\mathrm{d} - 3\mathrm{abc} + 2\mathrm{b}^3$ ($= f$ in 10th memoir),

4 $e = \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2$ ($= b$ in 10th memoir),

5 $f = \mathrm{a}^2\mathrm{f} - 5\mathrm{abe} + 2\mathrm{acd} + 8\mathrm{b}^2\mathrm{d} - 6\mathrm{bc}^2$,

6 $g = \mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2$,

7 $h = \mathrm{a}^2\mathrm{h} - 7\mathrm{abg} + 9\mathrm{acf} - 5\mathrm{ade} + 12\mathrm{b}^2\mathrm{f} - 30\mathrm{bce} + 20\mathrm{bd}^2$,

8 $i = \mathrm{ai} - 8\mathrm{bh} + 28\mathrm{cg} - 56\mathrm{df} + 35\mathrm{e}^2$,

&c.

(writing also (instead of d in tenth memoir)

$$\varepsilon = \mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} + 2\mathrm{bcd} - \mathrm{c}^3,$$

so that the equation $a^3d - a^2bc + 4c^3 - f^2 = 0$ of tenth memoir, is in the present notation $a^3\varepsilon - a^2ec + 4c^3 - d^2 = 0$); then the series of canonical forms is

Quadric $(1, 0, c\mathrm{)}(x, y)^2$,

Cubic $(1, 0, c, d\mathrm{)}(x, y)^3$,

Quartic $(1, 0, c, d, a^2e - 3c^2\mathrm{)}(x, y)^4$,

Quintic $(1, 0, c, d, a^2e - 3c^2, a^2f - 2cd\mathrm{)}(x, y)^5$,

&c.

the series of coefficients being

$1, 0, c, d,$	$a^2e + 1,$	$a^2f + 1,$	$a^4g + 1,$	$a^6i + 1,$
	$c^3 - 3$	$cd - 2$	$a^2ce - 15$	$a^2cf - 9$
			$c^3 + 45$	$a^2de + 5$
			$d^2 + 10$	$c^2d + 3$

$a^4h + 1,$	$a^6i + 1,$
$a^2cf - 9$	$a^4cg - 88$
$a^2de + 5$	$a^4e^2 - 35$
$c^2d + 3$	$a^2c^2e + 630$
	$a^2df + 56$
	$c^4 - 1575$
	$cd^2 - 392$

these values being in fact the expressions in terms of the seminvariants a, c, d, &c. of

1,	0,	ac,	a^2e,	a^3e,	a^4f,	a^5g,	a^6h,	a^7i
		$-b^2$,	$-3abc$,	$-4a^2bd$,	$-5a^3b^2e$,	$-6a^4bf$,	$-7a^5bg$,	$-8a^6bh$
			$+2b^3$,	$+6ab^2c$,	$+10a^2b^2d$,	$+15a^3b^2e$,	$+21a^4b^2f$,	$+28a^5b^2g$
				$-3b^4$,	$-10ab^3c$,	$-20a^2b^3d$,	$-35a^3b^3e$,	$-56a^4b^3f$
					$+4b^5$,	$+15ab^4c$,	$+35a^2b^4d$,	$+70a^5b^4e$
						$-5b^6$,	$-21ab^5c$,	$-56a^2b^5d$
							$+6b^7$,	$+28ab^6c$
								$-7b^8$

I annex verifications of the foregoing values:

	$a^2e = -3c^2$		
a^4e	1		$+1$
a^2bd	-4		-4
a^2c^2	$+3$	-3	0
ab^2c		$+6$	$+6$
b^4		-3	-3

	$a^2f =$	$-2cd$	
a^4f	$+1$		$+1$
a^3be	-5		-5
a^3cd	$+2$	-2	0
a^2b^2d	$+8$	$+2$	$+10$
a^2bc^2	-6	$+6$	0
ab^3c		-10	-10
b^5		$+4$	$+4$

	a^4g	$-15a^2ce$	$+45c^3$	$+10d^2$	
a^5g	$+1$				1
a^4bf	-6				-6
a^4ce	$+15$	-15			0
a^4d^2	-10			$+10$	0
a^3b^2e		$+15$			$+15$
a^3bcd		$+60$		-60	0
a^3c^3		-45	$+45$		0
a^2b^3d		-60		$+40$	-20
$a^2b^2c^2$		$+45$	-135	$+90$	0
ab^4c			$+135$	-120	$+15$
b^6			-45	$+40$	-5

	a^4h =	$-9a^2cf$	$+5a^2de$	$+3c^2d$	
a^6h	$+1$				$+1$
a^5bg	-7				-7
a^5cf	$+9$	-9			0
a^5de	-5		$+5$		0
a^4b^2f	$+12$	$+9$			$+21$
a^4bce	-30	$+45$	-15		0
a^4bd^2	$+20$		-20		0
a^4c^2d		-18	$+15$	$+3$	0
a^3b^3e		-45	$+10$		-35
a^3b^2cd		-54	$+60$	-6	0
a^3bc^3		$+54$	-45	-9	0
a^2b^4d		$+72$	-40	$+3$	$+35$
$a^2b^3c^2$		-54	$+30$	$+24$	0
ab^5c				-21	-21
b^7				$+6$	$+6$

	a^6i	$-28a^4cg$	$-35a^4e^2$	$+56a^2df$	$+630a^2c^2e^2$	$-1575b^4$	$-392c^2d$	=
a^7i	+ 1							+ 1
a^5bh	− 8							− 8
cg	+ 28	− 28						0
df	− 56				+ 56			0
e^2	+ 35		− 35					0
a^5b^2g		+ 28						+ 28
bcf		+ 168			− 168			0
bde			− 280		− 280			0
c^2e		− 420	− 210	+ 630				0
cd^2		+ 280			+ 112		− 392	0
a^4b^3f		− 168			+ 112			− 56
b^2ce		+ 420		− 1260	+ 840			0
b^2d^2		− 280	− 560		+ 448		− 392	0
bc^2d			+ 840	− 2520	− 672		+ 2352	0
c^4			− 315	+ 1890		− 1575		0
a^3b^4e				+ 630	− 560			+ 70
b^3cd				+ 5040	− 1120		− 3920	0
b^2c^3				− 3780	+ 1008	+ 6300	− 3528	0
a^2b^5d				− 2520	+ 896		+ 1568	− 56
b^4c^2				+ 1890	− 672	− 9450	+ 8232	0
$a\,b^6d$						+ 6300	− 6272	+ 28
a^0b^8						− 1575	+ 1568	− 7

It would be interesting to obtain the general law for the expressions of the canonical coefficients in terms of the seminvariants.

EXPRESSION FOR THE AREA OF A CONVEX QUADRILATERAL WHEN THE SUM OF TWO OPPOSITE ANGLES IS GIVEN.

By A. H. Anglin, B.A., F.R.S.E., M.R.I.A.

1. LET the quadrilateral be describable about a circle. If B be θ, and the sum of B and D which is given be 2ω, $D = 2\omega - \theta$ (fig. 14).

Then representing the area of the quadrilateral by Δ, we have, if a, b, c, d be the sides,

$$2\Delta = ab\sin\theta + ca\sin(2\omega - \theta),$$

or $$\Delta = ab\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta + cd\sin(\omega - \tfrac{1}{2}\theta)\cos(\omega - \tfrac{1}{2}\theta).$$

But $$a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos(2\omega - \theta);$$

therefore $$\sqrt{(ab)}\sin\tfrac{1}{2}\theta = \sqrt{(cd)}\sin(\omega - \tfrac{1}{2}\theta),$$

since $$(a - b)^2 = (c - d)^2;$$

therefore $ab \sin \frac{1}{2}\theta = \sqrt{(abcd)} \sin(\omega - \frac{1}{2}\theta)$,

and $cd \sin(\omega - \frac{1}{2}\theta) = \sqrt{(abcd)} \sin \frac{1}{2}\theta$.

Hence $\Delta = \sqrt{(abcd)} \{\sin(\omega - \frac{1}{2}\theta) \cos \frac{1}{2}\theta + \cos(\omega - \frac{1}{2}\theta) \sin \frac{1}{2}\theta\}$

$= \sqrt{(abcd)} \sin \omega$.

2. The area of any convex quadrilateral, in which the sum of two opposite angles is 2ω, is given by the equation

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \omega,$$

where $2s$ = sum of the sides.

For we have seen that

$$\Delta = ab \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta + cd \sin(\omega - \tfrac{1}{2}\theta) \cos(\omega - \tfrac{1}{2}\theta).$$

But $a^2 + b^2 - 2ab \cos\theta = c^2 + d^2 - 2cd \cos(2\omega - \theta)$;

therefore

$$(a+b)^2 - 4ab \cos^2 \tfrac{1}{2}\theta = (c-d)^2 + 4cd \sin^2(\omega - \tfrac{1}{2}\theta);$$

therefore

$$(a+b)^2 - (c-d)^2 = 4\{ab \cos^2 \tfrac{1}{2}\theta + cd \sin^2(\omega - \tfrac{1}{2}\theta),$$

and $(s-c)(s-d) = ab \cos^2 \frac{1}{2}\theta + cd \sin^2(\omega - \frac{1}{2}\theta)$.

Again

$$(a-b)^2 + 4ab \sin^2 \tfrac{1}{2}\theta = (c+d)^2 - 4cd \cos^2(\omega - \tfrac{1}{2}\theta);$$

therefore

$$(c+d)^2 - (a-b)^2 = 4\{ab \sin^2 \tfrac{1}{2}\theta + cd \cos^2(\omega - \tfrac{1}{2}\theta)\},$$

and $(s-a)(s-b) = ab \sin^2 \frac{1}{2}\theta + cd \cos^2(\omega - \frac{1}{2}\theta)$;

therefore $(s-a)(s-b)(s-c)(s-d)$

$= a^2b^2 \sin^2 \frac{1}{2}\theta \cos^2 \frac{1}{2}\theta + c^2d^2 \sin^2(\omega - \frac{1}{2}\theta) \cos^2(\omega - \frac{1}{2}\theta)$

$+ abcd\{\sin^2 \frac{1}{2}\theta \sin^2(\omega - \frac{1}{2}\theta) + \cos^2 \frac{1}{2}\theta \cos^2(\omega - \frac{1}{2}\theta)\}$

$= \{ab \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta + cd \sin(\omega - \frac{1}{2}\theta) \cos(\omega - \frac{1}{2}\theta)\}^2$

$+ abcd\{\sin \frac{1}{2}\theta \sin(\omega - \frac{1}{2}\theta) - \cos \frac{1}{2}\theta \cos(\omega - \frac{1}{2}\theta)\}^2$

$= \Delta^2 + abcd \cos^2 \omega$.

Hence $\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \omega$.

NOTE ON THE ENERGY OF STRAIN OF AN ISOTROPIC SOLID.

By H. T. Stearn, M.A., King's College, Cambridge.

THE process by which the expression for the internal energy, per unit volume, of a strained isotropic solid is obtained is usually one of considerable labour. The following method is shorter than any which I have seen.

Let a, b, c, s_1, s_2, s_3 be the components of strain at any point of the solid, *i.e.*, with the usual notation, let

$$a=\frac{du}{dx}, \quad 2s_1=\frac{dv}{dy}+\frac{dw}{dz}, \quad \&c.$$

Then the equation of the elongation quadric at the point is*

$$a\xi^2+b\eta^2+c\zeta^2+2s_1\eta\zeta+2s_2\zeta\xi+2s_3\xi\eta=\lambda.$$

Now for any change of axes the new values of these quantities are expressible in terms of the old and of the nine direction cosines of the new axes, and there are six relations amongst these nine direction cosines.

It follows that there must be three *and only three* independent relations between the new and old values of a, b, c, s_1, s_2, s_3.

But these we know to be

$$a+b+c=a'+b'+c',$$

$$bc+ca+ab-s_1^2-s_2^2-s_3^2=b'c'+c'a'+a'b'-s'_1{}^2-s'_2{}^2-s'_3{}^2,$$

$$abc-as_1^2-bs_2^2-cs_3^2+2s_1s_2s_3=a'b'c'-a's'_1{}^2-b's'_1{}^2-c'_1s'_3{}^2+2s'_1s'_2s'_3.$$

Now Φ, the internal energy per unit volume, may be shewn to be necessarily a homogeneous quadratic function of the components of strain. In the case of an isotropic solid, Φ must be an invariant quadratic function, and therefore, by the equations above, the only possible form of Φ is

$$\Phi=k(a+b+c)^2+k'(bc+ca+ab-s_1^2-s_2^2-s_3^2),$$

where k and k' are constants.

Writing $$k+\tfrac{1}{2}k'=\tfrac{1}{2}\lambda,$$
$$k'=-2\mu,$$

we have the ordinary form

$$\Phi=\tfrac{1}{2}\lambda(a+b+c)^2+\mu(a^2+b^2+c^2)+2\mu(s_1^2+s_2^2+s_3^2).$$

---

* Minchin's *Statics*, *Analysis of Strains*.

NOTE ON THE TIME OF DESCENT DOWN THE ARC OF A VERTICAL PARABOLA.

By J. W. L. Glaisher.

§ 1. IN a vertical parabola with vertex downwards if a particle start from rest at a height h above the vertex, and if the equation of the parabola be $y^2 = 4ax$, we have

$$\left(\frac{ds}{dt}\right)^2 = 2g(h-x), \quad \left(\frac{dx}{ds}\right)^2 = \frac{x}{a+x},$$

whence

$$\left(\frac{dx}{dt}\right)^2 = \frac{2gx(h-x)}{a+x},$$

and therefore, if t be the time of falling from the initial height h to the height x,

$$\sqrt{(2g)}.t = \int_x^h \sqrt{\left\{\frac{a+x}{x(h-x)}\right\}}\, dx.$$

Transforming this integral by putting $x = hu^2$, we have

$$\sqrt{(2g)}.t = 2\int_{\sqrt{\frac{x}{h}}}^1 \frac{\sqrt{(a+hu^2)}}{\sqrt{(1-u^2)}}\, du,$$

and, taking $u = \cos\phi$,

$$\sqrt{(2g)}.t = 2\int_0^{\arccos\sqrt{\frac{x}{h}}} \sqrt{(a+h\cos^2\phi)}\, d\phi$$

$$= 2\sqrt{(a+h)}.\int_0^{\arccos\sqrt{\frac{x}{h}}} \sqrt{\left(1-\frac{h}{a+h}\sin^2\phi\right)}\, d\phi,$$

whence

$$t = \sqrt{\left\{\frac{2(a+h)}{g}\right\}}\, E\left(\arccos\sqrt{\frac{x}{h}}\right), \quad \text{mod.} = \sqrt{\left(\frac{h}{a+h}\right)},$$

where E denotes Legendre's second elliptic integral.

If T denotes the time of descent to the vertex,

$$T=\sqrt{\left\{\frac{2(a+h)}{g}\right\}}E^1\left\{\sqrt{\left(\frac{h}{a+h}\right)}\right\},$$

where, following Legendre's notation, $E^1(k)$ denotes the complete integral to modulus k.

If T_1 denotes the time of falling freely from the height h,

$$T_1=\sqrt{\left(\frac{2h}{g}\right)},$$

and therefore

$$\frac{T}{T_1}=\sqrt{\left(\frac{a+h}{h}\right)}E^1\left\{\sqrt{\left(\frac{h}{a+h}\right)}\right\}$$
$$=\frac{E^1(k)}{k},\quad \text{where}\quad k=\sqrt{\left(\frac{h}{a+h}\right)}.$$

§ 2. In vol. XI. pp. 88–94 of *Grunert's Archiv*, Dr. Dienger considered the preceding question and expressed the result in terms of elliptic integrals, but in an extremely complicated form. In his investigation the modulus

$$=\frac{\sqrt{(h+a)}-\sqrt{a}}{\sqrt{(h+a)}+\sqrt{a}},$$

and if we denote this expression by λ, we have

$$\lambda=\frac{1-k'}{1+k'},$$

so that λ and k are connected as in Landen's transformation. Dr. Dienger's results should agree with those obtained in § 1, when the modulus is transformed to λ. This however does not appear to be the case: for, considering the time of descent to the vertex, we find from § 1, by transformation,

$$T=\sqrt{\left\{\frac{2(a+h)}{g}\right\}}\,\frac{2E^1(\lambda)-\lambda'^2F^1(\lambda)}{1+\lambda}.$$

Dr. Dienger works out the special case of $h=8a$, which gives $\lambda=\frac{1}{2}$. Putting $\lambda=\frac{1}{2}$ in this formula, we find

$$T=\sqrt{\left(\frac{a}{2g}\right)}\cdot\{8E^1(\tfrac{1}{2})-3F^1(\tfrac{1}{2})\},$$

and

$$\frac{T}{T_1}=\frac{8E^1(\frac{1}{2})-3F^1(\frac{1}{2})}{4\sqrt{2}}=1.18 \text{ nearly};$$

but Dr. Dienger's formula gives

$$T = 3\sqrt{\left(\frac{a}{2g}\right)}\{8E^1(\tfrac{1}{2}) - F^1(\tfrac{1}{2})\},$$

and he deduces from it that $\frac{T}{T_1}$ is equal to nearly $5\frac{1}{2}$, which evidently seems considerably too large.

§ 3. It may be mentioned, that in the course of his work Dr. Dienger evaluates the integrals

$$\int \frac{d\phi}{(1 - k\sin\phi)\,\Delta\phi} \quad \text{and} \quad \int \frac{\sin\phi d\phi}{(1 - k\sin\phi)\,\Delta\phi}.$$

Putting $\phi = \operatorname{am} u$, the first of these integrals

$$= \int \frac{du}{1 - k\,\operatorname{sn} u}$$

$$= \int \frac{1 + k\,\operatorname{sn} u}{\operatorname{dn}^2 u}\,du$$

$$= \int \frac{du}{\operatorname{dn}^2 u} + k\int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u}\,du.$$

Employing Jacobi's notation

$$\int \frac{du}{\operatorname{dn}^2 u} = \frac{1}{k'^2}\left\{\frac{E}{K}u + Z(u) - k^2\frac{\operatorname{sn} u\,\operatorname{cn} u}{\operatorname{dn} u}\right\},$$

$$\int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u}\,du = -\frac{1}{k'^2}\frac{\operatorname{cn} u}{\operatorname{dn} u},$$

and therefore

$$\int \frac{du}{1 - k\,\operatorname{sn} u} = \frac{1}{k'^2}\left\{\frac{E}{K}u + Z(u) - \frac{k\,\operatorname{cn} u\,(1 + k\,\operatorname{sn} u)}{\operatorname{dn} u}\right\};$$

also

$$\int \frac{\operatorname{sn} u\,du}{1 - k\,\operatorname{sn} u} = \frac{1}{k}\int \frac{du}{1 - k\,\operatorname{sn} u} - \frac{u}{k}$$

$$= \frac{1}{kk^2}\left\{\frac{E - k'^2K}{K}u + Z(u) - \frac{k\,\operatorname{cn} u\,(1 + k\,\operatorname{sn} u)}{\operatorname{dn} u}\right\}.$$

Taking the integrals between the limits 0 and K, we find

$$\int_0^K \frac{du}{1-k\,\mathrm{sn}\,u} = \frac{E+k}{k'^2},$$

$$\int_0^K \frac{\mathrm{sn}\,u\,du}{1-k\,\mathrm{sn}\,u} = \frac{E-k'^2K+k}{kk'^2}.$$

These results are in accordance with those given by Dr. Dienger, except that he puts

$$\int_0^\phi \frac{d\phi}{\Delta^3\phi} = \frac{E(\phi)}{k^2} - \frac{k^2\sin\phi\cos\phi}{k'^2\Delta\phi},$$

the true value of the first term on the right-hand side of the equation being $\frac{E(\phi)}{k'^2}$. This error renders the values of the integrals inaccurate, and is probably the cause of the final result being erroneous.

§ 4. If the parabola be vertical with its vertex upwards, and if the particle be projected along the curve from the vertex with the velocity due to a fall from a height c, then

$$\left(\frac{ds}{dt}\right)^2 = 2g(c+x), \quad \left(\frac{dx}{ds}\right)^2 = \frac{x}{a+x},$$

whence

$$\left(\frac{dx}{dt}\right)^2 = \frac{2gx(c+x)}{a+x}.$$

If then t be the time of falling through a depth x, then

$$\sqrt{(2g)}\,.\,t = \int_0^x \sqrt{\left\{\frac{a+x}{x(c+x)}\right\}}\,dx.$$

Putting $x = c\tan^2\phi$, we find

$$\sqrt{(2g)}\,.\,t = 2\int_0^{\arctan\sqrt{\frac{x}{c}}} \sqrt{(a\cos^2\phi + c\sin^2\phi)}\,d\phi.$$

If $a > c$, we have

$$\sqrt{(2g)}\,.\,t = 2\int_0^{\arctan\sqrt{\frac{x}{c}}} \sqrt{\{a-(a-c)\sin^2\phi\}}\,d\phi$$

$$= 2\sqrt{a}E\left(\arctan\sqrt{\frac{x}{c}}\right), \quad \mathrm{mod.} = \sqrt{\left(\frac{a-c}{a}\right)}.$$

If $c > a$, we have

$$\sqrt{(2g)}.t = 2\int_0^{\text{arc tan }\sqrt{\frac{x}{c}}} \sqrt{\{c-(c-a)\cos^2\phi\}}\, d\phi$$

$$= 2\sqrt{c}\int_{\text{arc tan }\sqrt{\frac{c}{x}}}^{\frac{1}{2}\pi} \sqrt{\left\{1-\frac{c-a}{c}\sin^2\phi\right\}}\, d\phi$$

$$= 2\sqrt{c}\left\{E' - E\left(\text{arc tan}\sqrt{\frac{c}{x}}\right)\right\}, \quad \text{mod.} = \sqrt{\left(\frac{c-a}{c}\right)}.$$

Thus

$$t = \sqrt{\left(\frac{2a}{g}\right)} E\left(\text{arc tan}\sqrt{\frac{x}{c}}\right), \quad \text{mod.} = \sqrt{\left(\frac{a-c}{a}\right)},$$

if $a > c$, and

$$= \sqrt{\left(\frac{2c}{g}\right)}\left\{E' - E\left(\text{arc tan}\sqrt{\frac{c}{x}}\right)\right\}, \quad \text{mod.} = \sqrt{\left(\frac{c-a}{c}\right)},$$

if $c > a$.

The results in this note cannot be new; but I have thought them worth giving, as the reduction to elliptic integrals is so very simple. In the earlier editions of Tait and Steele's *Dynamics*, the motion of a particle on a parabola with vertex upwards is considered, but the integral is not reduced to its elliptic form. This article is however omitted in the last (1878) edition.

A SYSTEM OF INTEGRALS INVOLVING ELLIPTIC FUNCTIONS.

By J. W. L. Glaisher.

Introduction, § 1.

§ 1. IN connexion with the integrals which occur in § 3 of the preceding note I have worked out the complete system of integrals to which they belong. These results are contained in the present note.

In the following lists the function $Z(u)$ is Jacobi's Zeta-function, defined by the equation

$$Z(u) = \int_0^u \mathrm{dn}^2 u\, du - \frac{E}{K} u$$

$$= \frac{K-E}{K}\, u - \int_0^u k^2 \,\mathrm{sn}^2 u\, du,$$

so that

$$Z(u) = \frac{\Theta'(u)}{\Theta(u)}.$$

The letters J and G, which are used for the sake of brevity, are defined by the equations

$$J = K - E,$$

$$G = E - k'^2 K;$$

so that we have

$$k^2 \int_0^K \mathrm{sn}^2 u\, du = J, \quad k^2 \int_0^K \mathrm{cn}^2 u\, du = G, \quad \int_0^K \mathrm{dn}^2 u\, du = E.$$

Hermite uses a different Zeta-function, and if this function is denoted by $Z_1(u)$, then

$$Z_1(u) = \int_0^u k^2 \,\mathrm{sn}^2 u\, du,$$

so that

$$-\frac{J}{K} u + Z(u) = \quad -Z_1(u),$$

$$\frac{G}{K} u + Z(u) = k'^2 u - Z_1(u),$$

$$\frac{E}{K} u + Z(u) = \quad u - Z_1(u).$$

Lists of indefinite integrals, §§ 2, 3.

§ 2.

(1) $$k'^2 \int \frac{1}{1-\mathrm{sn}\, u}\, du = -\frac{G}{K} u - Z(u) + \frac{\mathrm{dn}\, u\,(1+\mathrm{sn}\, u)}{\mathrm{cn}\, u},$$

(2) $$k'^2 \int \frac{1}{1+\mathrm{sn}\, u}\, du = -\frac{G}{K} u - Z(u) - \frac{\mathrm{dn}\, u\,(1-\mathrm{sn}\, u)}{\mathrm{cn}\, u},$$

(3) $$k'^2\int\frac{1}{1-k\,\mathrm{sn}\,u}\,du = \frac{E}{K}u+Z(u)-\frac{k\,\mathrm{cn}\,u\,(1+k\,\mathrm{sn}\,u)}{\mathrm{dn}\,u},$$

(4) $$k'^2\int\frac{1}{1+k\,\mathrm{sn}\,u}\,du = \frac{E}{K}u+Z(u)+\frac{k\,\mathrm{cn}\,u\,(1-k\,\mathrm{sn}\,u)}{\mathrm{dn}\,u},$$

(5) $$\int\frac{1}{1-\mathrm{cn}\,u}\,du = \frac{J}{K}u-Z(u)-\frac{\mathrm{dn}\,u\,(1+\mathrm{cn}\,u)}{\mathrm{sn}\,u},$$

(6) $$\int\frac{1}{1+\mathrm{cn}\,u}\,du = \frac{J}{K}u-Z(u)+\frac{\mathrm{dn}\,u\,(1-\mathrm{cn}\,u)}{\mathrm{sn}\,u},$$

(7) $$k^2\int\frac{1}{1-\mathrm{dn}\,u}\,du = \frac{J}{K}u-Z(u)-\frac{\mathrm{cn}\,u\,(1+\mathrm{dn}\,u)}{\mathrm{sn}\,u},$$

(8) $$k^2\int\frac{1}{1+\mathrm{dn}\,u}\,du = \frac{J}{K}u-Z(u)+\frac{\mathrm{cn}\,u\,(1-\mathrm{dn}\,u)}{\mathrm{sn}\,u},$$

(9) $$k^2k'\int\frac{1}{\mathrm{dn}\,u-k'}\,du = -\frac{G}{K}u-Z(u)+\frac{\mathrm{sn}\,u\,(\mathrm{dn}\,u+k')}{\mathrm{cn}\,u},$$

(10) $$k^2k'\int\frac{1}{\mathrm{dn}\,u+k'}\,du = \frac{G}{K}u+Z(u)-\frac{\mathrm{sn}\,u\,(\mathrm{dn}\,u-k')}{\mathrm{cn}\,u},$$

(11) $$k'^2\int\frac{\mathrm{sn}\,u}{1-\mathrm{sn}\,u}\,du = -\frac{E}{K}u-Z(u)+\frac{\mathrm{dn}\,u\,(1+\mathrm{sn}\,u)}{\mathrm{cn}\,u},$$

(12) $$k'^2\int\frac{\mathrm{sn}\,u}{1+\mathrm{sn}\,u}\,du = \frac{E}{K}u+Z(u)+\frac{\mathrm{dn}\,u\,(1-\mathrm{sn}\,u)}{\mathrm{cn}\,u},$$

(13) $$kk'^2\int\frac{\mathrm{sn}\,u}{1-k\,\mathrm{sn}\,u}\,du = \frac{G}{K}u+Z(u)-\frac{k\,\mathrm{cn}\,u\,(1+k\,\mathrm{sn}\,u)}{\mathrm{dn}\,u},$$

(14) $$kk'^2\int\frac{\mathrm{sn}\,u}{1+k\,\mathrm{sn}\,u}\,du = -\frac{G}{K}u-Z(u)-\frac{k\,\mathrm{cn}\,u\,(1-k\,\mathrm{sn}\,u)}{\mathrm{dn}\,u},$$

(15) $$\int\frac{\mathrm{cn}\,u}{1-\mathrm{cn}\,u}\,du = -\frac{E}{K}u-Z(u)-\frac{\mathrm{dn}\,u\,(1+\mathrm{cn}\,u)}{\mathrm{sn}\,u},$$

(16) $$\int\frac{\mathrm{cn}\,u}{1+\mathrm{cn}\,u}\,du = \frac{E}{K}u+Z(u)-\frac{\mathrm{dn}\,u\,(1-\mathrm{cn}\,u)}{\mathrm{sn}\,u},$$

(17) $$k^2\int\frac{\mathrm{dn}\,u}{1-\mathrm{dn}\,u}\,du = -\frac{G}{K}u-Z(u)-\frac{\mathrm{cn}\,u\,(1+\mathrm{dn}\,u)}{\mathrm{sn}\,u},$$

(18) $$k^2 \int \frac{\mathrm{dn}\,u}{1+\mathrm{dn}\,u}\,du = \frac{G}{K}u + Z(u) - \frac{\mathrm{cn}\,u\,(1-\mathrm{dn}\,u)}{\mathrm{sn}\,u},$$

(19) $$k^2 \int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u-k'}\,du = \frac{J}{K}u - Z(u) + \frac{\mathrm{sn}\,u\,(\mathrm{dn}\,u+k')}{\mathrm{cn}\,u},$$

(20) $$k^2 \int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u+k'}\,du = \frac{J}{K}u - Z(u) + \frac{\mathrm{sn}\,u\,(\mathrm{dn}\,u-k')}{\mathrm{cn}\,u},$$

(21) $$kk'^2 \int \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u-k\,\mathrm{cn}\,u}\,du = \frac{G}{K}u + Z(u) + k\,\mathrm{sn}\,u,$$

(22) $$kk'^2 \int \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u+k\,\mathrm{cn}\,u}\,du = -\frac{G}{K}u - Z(u) + k\,\mathrm{sn}\,u,$$

(23) $$k'^2 \int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u-k\,\mathrm{cn}\,u}\,du = \frac{E}{K}u + Z(u) + k\,\mathrm{sn}\,u,$$

(24) $$k'^2 \int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u+k\,\mathrm{cn}\,u}\,du = \frac{E}{K}u + Z(u) - k\,\mathrm{sn}\,u,$$

(25) $$k'^2 \int \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u-\mathrm{cn}\,u}\,du = -\frac{E}{K}u - Z(u) - \frac{1+\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u},$$

(26) $$k'^2 \int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u-\mathrm{cn}\,u}\,du = -\frac{G}{K}u - Z(u) - \frac{1+\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u},$$

(27) $$k'^2 \int \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u+\mathrm{cn}\,u}\,du = \frac{E}{K}u + Z(u) - \frac{1-\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u},$$

(28) $$k'^2 \int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u+\mathrm{cn}\,u}\,du = -\frac{G}{K}u - Z(u) + \frac{1-\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u},$$

(29) $$k' \int \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u-k'\,\mathrm{sn}\,u}\,du = -\frac{E}{K}u - Z(u) + \frac{k'+\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u},$$

(30) $$\int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u-k'\,\mathrm{sn}\,u}\,du = \frac{J}{K}u - Z(u) + \frac{k'+\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u},$$

(31) $$k' \int \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u+k'\,\mathrm{sn}\,u}\,du = \frac{E}{K}u + Z(u) + \frac{k'-\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u},$$

(32) $$\int \frac{\mathrm{dn}\,u}{\mathrm{dn}\,u+k'\,\mathrm{sn}\,u}\,du = \frac{J}{K}u - Z(u) - \frac{k'-\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u},$$

(33) $$\int \frac{\mathrm{dn}u}{1-\mathrm{sn}u}\,du = \frac{1+\mathrm{sn}u}{\mathrm{cn}u},$$

(34) $$\int \frac{\mathrm{dn}u}{1+\mathrm{sn}u}\,du = -\frac{1-\mathrm{sn}u}{\mathrm{cn}u},$$

(35) $$k\int \frac{\mathrm{cn}u}{1-k\,\mathrm{sn}u}\,du = \frac{1+k\,\mathrm{sn}u}{\mathrm{dn}u},$$

(36) $$k\int \frac{\mathrm{cn}u}{1+k\,\mathrm{sn}u}\,du = -\frac{1-k\,\mathrm{sn}u}{\mathrm{dn}u},$$

(37) $$\int \frac{\mathrm{dn}u}{1-\mathrm{cn}u}\,du = -\frac{1+\mathrm{cn}u}{\mathrm{sn}u},$$

(38) $$\int \frac{\mathrm{dn}u}{1+\mathrm{cn}u}\,du = \frac{1-\mathrm{cn}u}{\mathrm{sn}u},$$

(39) $$k^2\int \frac{\mathrm{cn}u}{1-\mathrm{dn}u}\,du = -\frac{1+\mathrm{dn}u}{\mathrm{sn}u},$$

(40) $$k^2\int \frac{\mathrm{cn}u}{1+\mathrm{dn}u}\,du = \frac{1-\mathrm{dn}u}{\mathrm{sn}u},$$

(41) $$k^2k'\int \frac{\mathrm{sn}u}{\mathrm{dn}u-k'}\,du = \frac{\mathrm{dn}u+k'}{\mathrm{cn}u},$$

(42) $$k^2k'\int \frac{\mathrm{sn}u}{\mathrm{dn}u+k'}\,du = -\frac{\mathrm{dn}u-k'}{\mathrm{cn}u},$$

(43) $$kk'^2\int \frac{\mathrm{sn}u}{\mathrm{dn}u-k\,\mathrm{cn}u}\,du = -(\mathrm{dn}u+k\,\mathrm{cn}u),$$

(44) $$kk'^2\int \frac{\mathrm{sn}u}{\mathrm{dn}u+k\,\mathrm{cn}u}\,du = \mathrm{dn}u-k\,\mathrm{cn}u,$$

(45) $$k'\int \frac{\mathrm{cn}u}{1-\mathrm{sn}u}\,du = \log\frac{\mathrm{dn}u+k'\,\mathrm{sn}u}{\mathrm{dn}u-k'} = \log\frac{\mathrm{dn}u+k'}{\mathrm{dn}u-k'\,\mathrm{sn}u},$$

(46) $$k'\int \frac{\mathrm{cn}u}{1+\mathrm{sn}u}\,du = \log\frac{\mathrm{dn}u+k'\,\mathrm{sn}u}{\mathrm{dn}u+k'} = \log\frac{\mathrm{dn}u-k'}{\mathrm{dn}u-k'\,\mathrm{sn}u},$$

(47) $$k'\int \frac{\mathrm{dn}u}{1-k\,\mathrm{sn}u}\,du = \arccos\left(\frac{k'\,\mathrm{cn}u}{1-k\,\mathrm{sn}u}\right) = \arctan\left(\frac{\mathrm{sn}u-k}{k'\,\mathrm{cn}u}\right),$$

(48) $$k'\int \frac{\mathrm{dn}u}{1+k\,\mathrm{sn}u}\,du = \arccos\left(\frac{k'\,\mathrm{cn}u}{1+k\,\mathrm{sn}u}\right) = \arctan\left(\frac{\mathrm{sn}u+k}{k'\,\mathrm{cn}u}\right),$$

(49) $$\int \frac{\operatorname{sn} u}{1-\operatorname{cn} u}\,du = \log\frac{1-\operatorname{dn} u}{\operatorname{dn} u+\operatorname{cn} u} = \log\frac{\operatorname{dn} u-\operatorname{cn} u}{1+\operatorname{dn} u} = 2\log\frac{\operatorname{sn}\frac{1}{2}u}{\operatorname{dn}\frac{1}{2}u},$$

(50) $$\int \frac{\operatorname{sn} u}{1+\operatorname{cn} u}\,du = \log\frac{1+\operatorname{dn} u}{\operatorname{dn} u+\operatorname{cn} u} = \log\frac{\operatorname{dn} u-\operatorname{cn} u}{1-\operatorname{dn} u} = -2\log\operatorname{cn}\tfrac{1}{2}u,$$

(51) $$k^2\int \frac{\operatorname{sn} u}{1-\operatorname{dn} u}\,du = \log\frac{1-\operatorname{cn} u}{\operatorname{dn} u+\operatorname{cn} u} = \log\frac{\operatorname{dn} u-\operatorname{cn} u}{1+\operatorname{cn} u} = 2\log\frac{\operatorname{sn}\frac{1}{2}u}{\operatorname{cn}\frac{1}{2}u},$$

(52) $$k^2\int \frac{\operatorname{sn} u}{1+\operatorname{dn} u}\,du = \log\frac{1+\operatorname{cn} u}{\operatorname{dn} u+\operatorname{cn} u} = \log\frac{\operatorname{dn} u-\operatorname{cn} u}{1-\operatorname{cn} u} = -2\log\operatorname{dn}\tfrac{1}{2}u,$$

(53) $$k^2\int \frac{\operatorname{cn} u}{\operatorname{dn} u-k'}\,du = \log\frac{1+\operatorname{sn} u}{\operatorname{dn} u-k'\operatorname{sn} u} = \log\frac{\operatorname{dn} u+k'\operatorname{sn} u}{1-\operatorname{sn} u},$$

(54) $$k^2\int \frac{\operatorname{cn} u}{\operatorname{dn} u+k'}\,du = \log\frac{1+\operatorname{sn} u}{\operatorname{dn} u+k'\operatorname{sn} u} = \log\frac{\operatorname{dn} u-k'\operatorname{sn} u}{1-\operatorname{sn} u},$$

(55) $$k'^2\int \frac{1}{\operatorname{dn} u-k\operatorname{cn} u}\,du = \arcsin\left(\frac{k'^2\operatorname{sn} u}{\operatorname{dn} u-k\operatorname{cn} u}\right) = \arctan\left(\frac{k'^2\operatorname{sn} u}{\operatorname{cn} u-k\operatorname{dn} u}\right),$$

(56) $$k'^2\int \frac{1}{\operatorname{dn} u+k\operatorname{cn} u}\,du = \arcsin\left(\frac{k'^2\operatorname{sn} u}{\operatorname{dn} u+k\operatorname{cn} u}\right) = \arctan\left(\frac{k'^2\operatorname{sn} u}{\operatorname{cn} u+k\operatorname{dn} u}\right),$$

(57) $$k'^2\int \frac{1}{\operatorname{dn} u-\operatorname{cn} u}\,du = -\frac{\operatorname{dn} u+\operatorname{cn} u}{\operatorname{sn} u},$$

(58) $$k'^2\int \frac{1}{\operatorname{dn} u+\operatorname{cn} u}\,du = \frac{\operatorname{dn} u-\operatorname{cn} u}{\operatorname{sn} u},$$

(59) $$k'\int \frac{1}{\operatorname{dn} u-k'\operatorname{sn} u}\,du = \frac{\operatorname{dn} u+k'\operatorname{sn} u}{\operatorname{cn} u},$$

(60) $$k'\int \frac{1}{\operatorname{dn} u+k'\operatorname{sn} u}\,du = -\frac{\operatorname{dn} u-k'\operatorname{sn} u}{\operatorname{cn} u},$$

(61) $$k'^2\int \frac{\operatorname{sn} u}{\operatorname{dn} u-\operatorname{cn} u}\,du = \log\frac{1-\operatorname{dn} u}{1+\operatorname{cn} u} = \log\frac{1-\operatorname{cn} u}{1+\operatorname{dn} u},$$

(62) $$k'^2\int \frac{\operatorname{sn} u}{\operatorname{dn} u+\operatorname{cn} u}\,du = \log\frac{1-\operatorname{cn} u}{1-\operatorname{dn} u} = \log\frac{1+\operatorname{dn} u}{1+\operatorname{cn} u},$$

(63) $$\int \frac{\operatorname{cn} u}{\operatorname{dn} u-k'\operatorname{sn} u}\,du = \log\frac{1+\operatorname{sn} u}{\operatorname{dn} u-k'} = \log\frac{\operatorname{dn} u+k'}{1-\operatorname{sn} u},$$

(64) $$\int \frac{\operatorname{cn} u}{\operatorname{dn} u+k'\operatorname{sn} u}\,du = \log\frac{1+\operatorname{sn} u}{\operatorname{dn} u+k'} = \log\frac{\operatorname{dn} u-k'}{1-\operatorname{sn} u}.$$

In (45)–(56) and (61)–(64), where two or more values are given of the integral, they are not necessarily equal, but may differ by a constant.

Half of these evaluations may be deduced from the other half by substituting $K+u$ for u and using the formulæ

$$\operatorname{sn}(K+u) = \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

$$\operatorname{cn}(K+u) = -k' \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

$$\operatorname{dn}(K+u) = k' \frac{1}{\operatorname{dn} u};$$

and in this manner the results were verified.

§ 3. In every case the value of the integral may be obtained by multiplying both numerator and denominator by the denominator with the sign connecting the two terms changed; thus, for example, taking (1), we have

$$k'^2 \int \frac{1}{1-\operatorname{sn} u}\, du = k'^2 \int \frac{1+\operatorname{sn} u}{\operatorname{cn}^2 u}\, du = k'^2 \int \frac{1}{\operatorname{cn}^2 u}\, du + k'^2 \int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u}\, au$$

$$= -\frac{G}{K} u - Z(u) + \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} + \frac{\operatorname{dn} u}{\operatorname{cn} u}.$$

The values of the integrals required for the evaluation of the integrals given in the preceding list are contained in the following systems of formulæ:

I.

$$k \int \operatorname{sn} u\, du = \log(\operatorname{dn} u - k \operatorname{cn} u),$$

$$k \int \operatorname{cn} u\, du = \arcsin(k \operatorname{sn} u),$$

$$\int \operatorname{dn} u\, du = \arcsin(\operatorname{sn} u),$$

$$\int \frac{1}{\mathrm{sn}u}\,du = \log\frac{\mathrm{dn}u - \mathrm{cn}u}{\mathrm{sn}u},$$

$$\int \frac{\mathrm{dn}u}{\mathrm{sn}u}\,du = \log\frac{1-\mathrm{cn}u}{\mathrm{sn}u},$$

$$\int \frac{\mathrm{cn}u}{\mathrm{sn}u}\,du = \log\frac{1-\mathrm{dn}u}{\mathrm{sn}u},$$

$$\int \frac{\mathrm{dn}u}{\mathrm{cn}u}\,du = \log\frac{1+\mathrm{sn}u}{\mathrm{cn}u},$$

$$k'\int \frac{1}{\mathrm{cn}u}\,du = \log\frac{\mathrm{dn}u + k'\,\mathrm{sn}u}{\mathrm{cn}u},$$

$$k'\int \frac{\mathrm{sn}u}{\mathrm{cn}u}\,du = \log\frac{\mathrm{dn}u + k'}{\mathrm{cn}u},$$

$$k\int \frac{\mathrm{cn}u}{\mathrm{dn}u}\,du = \log\frac{1+k\,\mathrm{sn}u}{\mathrm{dn}u},$$

$$kk'\int \frac{\mathrm{sn}u}{\mathrm{dn}u}\,du = \arcsin\left(\frac{k'}{\mathrm{dn}u}\right),$$

$$k'\int \frac{1}{\mathrm{dn}u}\,du = \arcsin\left(k'\,\frac{\mathrm{sn}u}{\mathrm{dn}u}\right).$$

II.

$$k^2\int \mathrm{sn}^2u\,du = \frac{J}{K}u - Z(u),$$

$$k^2\int \mathrm{cn}^2u\,du = \frac{G}{K}u + Z(u),$$

$$\int \mathrm{dn}^2u\,du = \frac{E}{K}u + Z(u),$$

$$\int \frac{1}{\mathrm{sn}^2u}\,du = \frac{J}{K}u - Z(u) - \frac{\mathrm{cn}u\,\mathrm{dn}u}{\mathrm{sn}u},$$

$$\int \frac{\mathrm{dn}^2u}{\mathrm{sn}^2u}\,du = -\frac{G}{K}u - Z(u) - \frac{\mathrm{cn}u\,\mathrm{dn}u}{\mathrm{sn}u},$$

$$\int \frac{\mathrm{cn}^2u}{\mathrm{sn}^2u}\,du = -\frac{E}{K}u - Z(u) - \frac{\mathrm{cn}u\,\mathrm{dn}u}{\mathrm{sn}u},$$

$$\int \frac{\mathrm{dn}^2 u}{\mathrm{cn}^2 u}\, du = \frac{J}{K} u - Z(u) + \frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}u},$$

$$k'^2 \int \frac{1}{\mathrm{cn}^2 u}\, du = -\frac{G}{K} u - Z(u) + \frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}u},$$

$$k'^2 \int \frac{\mathrm{sn}^2 u}{\mathrm{cn}^2 u}\, du = -\frac{E}{K} u - Z(u) + \frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}u},$$

$$k^2 \int \frac{\mathrm{cn}^2 u}{\mathrm{dn}^2 u}\, du = \frac{J}{K} u - Z(u) + \frac{k^2\,\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u},$$

$$k^2 k'^2 \int \frac{\mathrm{sn}^2 u}{\mathrm{dn}^2 u}\, du = \frac{G}{K} u + Z(u) - \frac{k^2\,\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u},$$

$$k'^2 \int \frac{1}{\mathrm{dn}^2 u}\, du = \frac{E}{K} u + Z(u) - \frac{k^2\,\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u}.$$

III.

$$\int \frac{\mathrm{cn}u}{\mathrm{sn}^2 u}\, du = -\frac{\mathrm{dn}u}{\mathrm{sn}u}, \qquad \int \frac{\mathrm{dn}u}{\mathrm{sn}^2 u}\, du = -\frac{\mathrm{cn}u}{\mathrm{sn}u},$$

$$k'^2 \int \frac{\mathrm{sn}u}{\mathrm{cn}^2 u}\, du = \frac{\mathrm{dn}u}{\mathrm{cn}u}, \qquad \int \frac{\mathrm{dn}u}{\mathrm{cn}^2 u}\, du = \frac{\mathrm{sn}u}{\mathrm{cn}u},$$

$$k'^2 \int \frac{\mathrm{sn}u}{\mathrm{dn}^2 u}\, du = -\frac{\mathrm{cn}u}{\mathrm{dn}u}, \qquad \int \frac{\mathrm{cn}u}{\mathrm{dn}^2 u}\, du = \frac{\mathrm{sn}u}{\mathrm{dn}u},$$

$$\int \frac{\mathrm{cn}u\,\mathrm{dn}u}{\mathrm{sn}^2 u}\, du = -\frac{1}{\mathrm{sn}u}, \qquad \int \mathrm{cn}u\,\mathrm{dn}u\, du = \mathrm{sn}u,$$

$$\int \frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}^2 u}\, du = \frac{1}{\mathrm{cn}u}, \qquad \int \mathrm{sn}u\,\mathrm{dn}u\, du = -\mathrm{cn}u,$$

$$k^2 \int \frac{\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}^2 u}\, du = \frac{1}{\mathrm{dn}u}, \qquad k^2 \int \mathrm{sn}u\,\mathrm{cn}u = -\mathrm{dn}u.$$

The systems I and II are taken from the *Messenger of Mathematics*, vol. XI. pp. 128, 129, where, however, they are given in a slightly different form.

List of definite integrals, §§ 4, 5.

§ 4. By integration between the limits 0 and K, we obtain from I, II, III in the last section the evaluations:

$$k\int_0^K \operatorname{sn} u\, du = k\int_0^K \frac{\operatorname{cn} u}{\operatorname{dn} u}\, du = \log\frac{1+k}{k'},$$

$$k\int_0^K \operatorname{cn} u\, du = kk'\int_0^K \frac{\operatorname{sn} u}{\operatorname{dn} u}\, du = \operatorname{arc}\sin k,$$

$$\int_0^K \operatorname{dn} u\, du = k'\int_0^K \frac{1}{\operatorname{dn} u}\, du = \tfrac{1}{2}\pi,$$

$$k^2\int_0^K \operatorname{sn}^2 u\, du = k^2\int_0^K \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}\, du = J,$$

$$k^2\int_0^K \operatorname{cn}^2 u\, du = k^2k'^2\int_0^K \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}\, du = G,$$

$$\int_0^K \operatorname{dn}^2 u\, du = k'^2\int_0^K \frac{1}{\operatorname{dn}^2 u}\, du = E,$$

$$k'^2\int_0^K \frac{\operatorname{sn} u}{\operatorname{dn}^2 u}\, du = 1, \qquad k'\int_0^K \frac{\operatorname{cn} u}{\operatorname{dn}^2 u}\, du = 1,$$

$$k^2k'\int_0^K \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^2 u}\, du = 1 - k'.$$

The pairs of integrals in the same line may be converted into one another by the substitution of $K - u$ for u, since

$$\int_0^K \phi(K-u)\, du = \int_0^K \phi(u)\, du$$

and

$$\operatorname{sn}(K-u) = \frac{\operatorname{cn} u}{\operatorname{dn} u}, \quad \operatorname{cn}(K-u) = k'\frac{\operatorname{sn} u}{\operatorname{dn} u}, \quad \operatorname{dn}(K-u) = k'\frac{1}{\operatorname{dn} u}.$$

§ 5. From these results, or directly from the formulæ in § 2, we may derive the following system of definite integrals taken between the limits 0 and K:

(1) $$k'^2 \int_0^K \frac{1}{1+\operatorname{sn} u}\, du = -G+1,$$

(2) $$k'^2 \int_0^K \frac{1}{1-k\operatorname{sn} u}\, du = E+k,$$

(3) $$k'^2 \int_0^K \frac{1}{1+k\operatorname{sn} u}\, du = E-k,$$

(4) $$\int_0^K \frac{1}{1+\operatorname{cn} u}\, du = J+k',$$

(5) $$k^2 \int_0^K \frac{1}{1+\operatorname{dn} u}\, du = J,$$

(6) $$k^2k' \int_0^K \frac{1}{\operatorname{dn} u+k'}\, du = G,$$

(7) $$k'^2 \int_0^K \frac{\operatorname{sn} u}{1+\operatorname{sn} u}\, du = E-1,$$

(8) $$kk'^2 \int_0^K \frac{\operatorname{sn} u}{1-k\operatorname{sn} u}\, du = G+k,$$

(9) $$kk'^2 \int_0^K \frac{\operatorname{sn} u}{1+k\operatorname{sn} u}\, du = -G+k,$$

(10) $$\int_0^K \frac{\operatorname{cn} u}{1+\operatorname{cn} u}\, du = E-k',$$

(11) $$k^2 \int_0^K \frac{\operatorname{dn} u}{1+\operatorname{dn} u}\, du = G,$$

(12) $$k^2 \int_0^K \frac{\operatorname{dn} u}{\operatorname{dn} u+k'}\, du = J,$$

$$(13)\quad kk'^2\int_0^K \frac{\operatorname{cn}u}{\operatorname{dn}u - k\operatorname{cn}u}\,du = \quad G + k,$$

$$(14)\quad kk'^2\int_0^K \frac{\operatorname{cn}u}{\operatorname{dn}u + k\operatorname{cn}u}\,du = -G + k,$$

$$(15)\quad k'^2\int_0^K \frac{\operatorname{dn}u}{\operatorname{dn}u - k\operatorname{cn}u}\,du = \quad E + k,$$

$$(16)\quad k'^2\int_0^K \frac{\operatorname{dn}u}{\operatorname{dn}u + k\operatorname{cn}u}\,du = \quad E - k,$$

$$(17)\quad k'^2\int_0^K \frac{\operatorname{cn}u}{\operatorname{dn}u + \operatorname{cn}u}\,du = \quad E - 1,$$

$$(18)\quad k'^2\int_0^K \frac{\operatorname{dn}u}{\operatorname{dn}u + \operatorname{cn}u}\,du = -G + 1,$$

$$(19)\quad k'\int_0^K \frac{\operatorname{sn}u}{\operatorname{dn}u + k'\operatorname{sn}u}\,du = \quad E - k',$$

$$(20)\quad \int_0^K \frac{\operatorname{dn}u}{\operatorname{dn}u + k'\operatorname{sn}u}\,du = \quad J + k',$$

$$(21)\quad \int_0^K \frac{\operatorname{dn}u}{1 + \operatorname{sn}u}\,du = \quad 1,$$

$$(22)\quad kk'\int_0^K \frac{\operatorname{cn}u}{1 - k\operatorname{sn}u}\,du = \quad 1 + k - k',$$

$$(23)\quad kk'\int_0^K \frac{\operatorname{cn}u}{1 + k\operatorname{sn}u}\,du = -1 + k + k',$$

$$(24)\quad \int_0^K \frac{\operatorname{dn}u}{1 + \operatorname{cn}u}\,du = \quad 1,$$

$$(25)\quad k^2\int_0^K \frac{\operatorname{cn}u}{1 + \operatorname{dn}u}\,du = 1 - k',$$

$$(26)\quad k^2k'\int_0^K \frac{\operatorname{sn}u}{\operatorname{dn}u + k'}\,du = 1 - k',$$

$$(27)\quad kk'^2\int_0^K \frac{\operatorname{sn}u}{\operatorname{dn}u - k\operatorname{cn}u}\,du = \quad 1 + k - k',$$

$$(28)\quad kk'^2\int_0^K \frac{\operatorname{sn}u}{\operatorname{dn}u + k\operatorname{cn}u}\,du = -1 + k + k',$$

$$(29)\quad k'\int_0^K \frac{\operatorname{cn} u}{1+\operatorname{sn} u}\,du = \log(1+k'),$$

$$(30)\quad k'\int_0^K \frac{\operatorname{dn} u}{1-k\operatorname{sn} u}\,du = \pi - \arccos k,$$

$$(31)\quad k'\int_0^K \frac{\operatorname{dn} u}{1+k\operatorname{sn} u}\,du = \arccos k,$$

$$(32)\quad \int_0^K \frac{\operatorname{sn} u}{1+\operatorname{cn} u}\,du = \log\frac{1+k'}{k'},$$

$$(33)\quad k^2\int_0^K \frac{\operatorname{sn} u}{1+\operatorname{dn} u}\,du = -\log k',$$

$$(34)\quad k^2\int_0^K \frac{\operatorname{cn} u}{\operatorname{dn} u+k'}\,du = -\log k',$$

$$(35)\quad k'^2\int_0^K \frac{1}{\operatorname{dn} u-k\operatorname{cn} u}\,du = \pi - \arccos k,$$

$$(36)\quad k'^2\int_0^K \frac{1}{\operatorname{dn} u+k\operatorname{cn} u}\,du = \arccos k,$$

$$(37)\quad k'\int_0^K \frac{1}{\operatorname{dn} u+k\operatorname{cn} u}\,du = 1,$$

$$(38)\quad k'\int_0^K \frac{1}{\operatorname{dn} u+k'\operatorname{sn} u}\,du = 1,$$

$$(39)\quad k'^2\int_0^K \frac{\operatorname{sn} u}{\operatorname{dn} u+\operatorname{cn} u}\,du = \log(1+k'),$$

$$(40)\quad \int_0^K \frac{\operatorname{cn} u}{\operatorname{dn} u+k'\operatorname{sn} u}\,du = \log\frac{1+k'}{k'}.$$

Half of these formulæ may be deduced from the other half by the substitution of $K-u$ for u, as in § 4.

ON THE DIFFERENTIAL EQUATION

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0.$$

By Captain P. A. MacMahon, R.A.

§ 1. IN the Comptes Rendus, t. LXVI. p. 1144, Allegret has considered the integral of the differential equation

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy = 0,$$

where
$$X = A + 3Bx + 3Cx^2 + Dx^3,$$
$$Y = A + 3By + 3Cy^2 + Dy^3,$$

and has obtained the result

$$A + \left(\frac{yX^{\frac{1}{3}} - xY^{\frac{1}{3}}}{x-y}\right)^3 + \alpha xy\left\{D - \left(\frac{X^{\frac{1}{3}} - Y^{\frac{1}{3}}}{x-y}\right)^3\right\} = 0,$$

where α is the arbitrary constant.

This equation is symmetrical in x, y, and α, and putting z for α, I have in the following solution obtained the rational algebraical integral in the form

$$XYZ = \{A + B(x+y+z) + C(yx + zx + xy) + Dxyz\}^3,$$

where
$$Z = A + 3Bx + 3Cz^2 + Dx^3,$$

and z is now the arbitrary constant.

Write the equation in the form

$$\frac{dx}{\{(x-\alpha)(x^2 - 2px + q)\}^{\frac{2}{3}}} + \frac{dy}{\{(y-\alpha)(y^2 - 2py + q)\}^{\frac{2}{3}}} = 0.$$

To reduce these to elliptic integrals, put

$$\frac{X}{(x-\alpha)^3} = x_1^3, \quad \frac{Y}{(y-\alpha)^3} = y_1^2;$$

then
$$\frac{dx_1}{\sqrt{(x_1^3 - b^3)}} + \frac{dy_1}{\sqrt{(y_1^3 - b^3)}} = 0,$$

$$b^3 = \frac{q - p^2}{\alpha^2 - 2p\alpha + q},$$

Euler's integral of which is

$$\left\{\frac{X_1 \pm Y_1}{x_1 - y_1}\right\}^2 = x_1 + y_1 + z_1;$$

in which $X_1 = \sqrt{(x_1^3 - b^3)}$, $Y_1 = \sqrt{(y_1^3 - b^3)}$,

and z_1 is the arbitrary constant.

It will be shown hereafter that z_1 is such that

$$\int_{x_1}^{\infty} \frac{dx_1}{X_1} + \int_{y_1}^{\infty} \frac{dy_1}{Y_1} + \int_{z_1}^{\infty} \frac{dz_1}{Z_1} = 0,$$

$$Z_1 = \sqrt{(z_1^3 - b^3)}.$$

To reduce these we put

$$x_1 = b + b\sqrt{3}\,\frac{1 + \operatorname{cn} u}{1 - \operatorname{cn} u}$$

$$y_1 = b + b\sqrt{3}\,\frac{1 + \operatorname{cn} v}{1 - \operatorname{cn} v},$$

$$z_1 = b + b\sqrt{3}\,\frac{1 + \operatorname{cn} w}{1 - \operatorname{cn} w},$$

and then $$u + v + w = 0.$$

Therefore when

$$x_1 = y_1, \quad z_1 \text{ is finite},$$
$$y_1 = z_1, \quad x_1 \text{ is finite},$$
$$z_1 = x_1, \quad y_1 \text{ is finite};$$

so that Euler's integral may be written in the forms

$$x_1 + y_1 + z_1 = \left\{\frac{X_1 - Y_1}{x_1 - y_1}\right\}^2,$$

$$= \left\{\frac{Y_1 - Z_1}{y_1 - z_1}\right\}^2,$$

$$= \left\{\frac{Z_1 - X_1}{z_1 - x_1}\right\}^2.$$

Therefore

$$(x_1 + y_1 + z_1)(x_1 - y_1)^2 = (X_1 - Y_1)^2,$$
$$(x_1 + y_1 + z_1)(y_1 - z_1)^2 = (Y_1 - Z_1)^2,$$
$$(x_1 + y_1 + z_1)(z_1 - x_1)^2 = (Z_1 - X_1)^2,$$

and adding

$$(x_1+y_1+z_1)(x_1^2+y_1^2+z_1^2-y_1z_1-z_1x_1-x_1y_1)$$
$$=X_1^2+Y_1^2+Z_1^2-Y_1Z_1-Z_1X_1-X_1Y_1;$$

or $$x_1^3+y_1^3+z_1^3-3x_1y_1z_1=x_1^3+y_1^3+z_1^3-3b^3$$
$$-Y_1Z_1-Z_1X_1-X_1Y_1,$$

that is $$3x_1y_1z_1=3b^3+Y_1Z_1+Z_1X_1+X_1Y_1,$$

an equation which, when both sides are cubed, is the complete integral of the differential equation

$$X^{-\frac{2}{3}}dx+Y^{-\frac{2}{3}}dy=0;$$

for putting $$z_1^3=\frac{z^2-2pz+q}{(x-\alpha)^2}$$ for symmetry,

and $$\alpha^2-2p\alpha+q=V,\quad \alpha-p=P,$$

we find
$$X_1=\frac{P(x-\alpha)+V}{\sqrt{V}.(x-\alpha)},$$
$$Y=\frac{P(y-\alpha)+V}{\sqrt{V}.(y-\alpha)},$$
$$Z_1=\frac{P(z-\alpha)+V}{\sqrt{V}.(z-\alpha)};$$

whence, substituting in the equation and cubing,

$$\frac{27XYZ}{(x-\alpha)^3(y-\alpha)^3(z-\alpha)^3}=\left\{3\,\frac{V-P^2}{V}+\frac{P(y-\alpha)+V}{\sqrt{V}.(y-\alpha)}\cdot\frac{P(z-\alpha)+V}{\sqrt{V}.(z-\alpha)}\right.$$
$$\left.+\frac{P(z-\alpha)+V}{\sqrt{V}.(2-\alpha)}\cdot\frac{P(x-\alpha)+V}{\sqrt{V}.(x-\alpha)}+\frac{P(x-\alpha)+V}{\sqrt{V}.(x-\alpha)}\cdot\frac{P(y-\alpha)+V}{\sqrt{V}.(y-\alpha)}\right\}^3;$$

or

$$27V^3XYZ=[3(V-P^2)(x-\alpha)(y-\alpha)(z-\alpha)$$
$$+(x-\alpha)\{P(y-\alpha)+V\}\{P(x-\alpha)+V\}$$
$$+(y-\alpha)\{P(z-\alpha)+V\}\{P(x-\alpha)+V\}$$
$$+(z-\alpha)\{P(x-\alpha)+V\}\{P(y-\alpha)+V\}]^3$$
$$=[3(V-P^2)(z-\alpha)\{xy-\alpha(x+y)+\alpha^2\}$$
$$+\{P(z-\alpha)+V\}\{2Pxy-(2P\alpha-V)(x+y)+2P\alpha^2-2\alpha V\}$$
$$+(z-\alpha)\{P^2xy-(\alpha P^2-PV)(x+y)+(\alpha P-V)^2\}]^3.$$

The coefficients of xy inside the bracket on the right-hand side of this equation is

$$3(V-P^2)(z-\alpha)+2P^2(z-\alpha)+2PV+P^2(z-\alpha)$$
$$= V\{3(z-\alpha)+2P\} = V\{3z-(2p+\alpha)\}$$
$$= 3V(z+C).$$

That of $(x+y)$ is

$$-3(V-P^2)\alpha(z-\alpha)-(2P\alpha-V)\{P(z-\alpha)+V\}$$
$$-P(z-\alpha)(\alpha P-V);$$
$$= -V(2p+\alpha)(z-\alpha)-V(\alpha^2-q)$$
$$= V\{-(2p+\alpha)z+2p\alpha+q\}$$
$$= 3V(B+Cz).$$

The term independent of x and y is

$$3\alpha^2(V-P^2)(z-\alpha)+\{P(z-\alpha)+V\}(2P\alpha^2-2\alpha V);$$
$$= V\{2p\alpha+q)z-3\alpha q\}$$
$$= 3V\{A+Bz\}.$$

Hence

$$XYZ=\{A+B(x+y+z)+C(yz+zx+xy)+xyz\}^3,$$

or, replacing the coefficient D, we have

$$XYZ=\{A+B(x+y+z)+C(yz+zx+xy)+Dxyz\}^3,$$

as the rational integral of

$$\frac{dx}{\{A+3Bx+3Cx^2+Dx^3\}^{\frac{2}{3}}}+\frac{dy}{\{A+3By+3Cy^2+Dy^3\}}=0.$$

This equation is, when expanded, of the form:

$$\mathrm{a}+\mathrm{h}y+\mathrm{g}y^2+\mathrm{p}y^3$$
$$+x(\mathrm{h}+\mathrm{b}y+\mathrm{f}y^2+\mathrm{q}y^3)$$
$$+x^2(\mathrm{g}+\mathrm{f}y+\mathrm{c}y^2+\mathrm{r}y^3)$$
$$+x^3(\mathrm{p}+\mathrm{q}y+\mathrm{r}y^2+\mathrm{d}y^3)=0.$$

The coefficients, of which ten are different, are therefore derivable from a matrix of the form of the determinant

which expresses the discriminant of the general quadric surface:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0$$

(compare Cayley, *Elliptic Functions*, ch. XIV., p. 340).

This shews that the complete integral of the differential equation is an equation $u = 0$, where u is a symmetric cubi-cubic function of (x, y); that is, a symmetric function, cubic in regard to each variable separately.

It is also a particular integral of the differential equation

$$X^{-\frac{2}{3}}\,dx + Y^{-\frac{2}{3}}\,dy + Z^{-\frac{2}{3}}\,dz = 0,$$

in a form, symmetrical in (x, y, z).

To prove the fact stated above, that the constant in Euler's integral is equal to z, we have only to shew that the equation

$$\left\{\frac{\sqrt{(x_1^3 - b^3)} \pm \sqrt{(y_1^3 - b^3)}}{x_1 - y_1}\right\} = x_1 + y_1 + z_1,$$

is symmetrical in x_1, y_1 and z_1.

Rationalising we find

$$\{-b^3 - \tfrac{1}{2}(x_1 - y_1)^2 z_1 + \tfrac{1}{2}x_1 y_1 (x_1 + y_1)\}^2 = (x_1^3 - b^3)(y_1^3 - b^3);$$

therefore

$$\tfrac{1}{4}(x_1 - y_1)^4 z_1^2 + z_1 (x_1 - y_1)\{b^3 - \tfrac{1}{2}x_1 y_1 (x_1 + y_1)\} + b^3 (x_1 + y_1)(x_1 - y_1)^2 + \tfrac{1}{4}x_1^2 y_1^2 (x_1 - y_1)^2 = 0.$$

Rejecting the factor $(x_1 - y_1)^2$ we find

$$(x_1 - y_1)^2 z_1^2 + 4b^3 z_1 - 2x_1 y_1 z_1 (x_1 + y_1) + 4b^3 (x_1 + y_1) + x_1^2 y_1^2 = 0,$$

or

$$y_1^2 z_1^2 + z_1^2 x_1^2 + x_1^2 y_1^2 = 2(x_1 y_1 z_1 - 2b^3)(x_1 + y_1 + z_1),$$

which is symmetrical in x_1, y_1 and z_1, and therefore z_1 is such that

$$\frac{dx_1}{\sqrt{(x_1^3 - b^3)}} + \frac{dy_1}{\sqrt{(y_1^3 - b^3)}} + \frac{dz_1}{\sqrt{(z_1^3 - b^3)}} = 0.$$

The coefficients, ten in number, of the developed equation are as follows:

where $a = B^2 - AC$; $b = AD - BC$; $c = C^2 - BD$,

$$\mathrm{a} = -3Aaz^2 + (Ab - Ba)\,z^3,$$

$$\mathrm{h} = 3\{Aaz - Baz^2 + Bbz^3\},$$

$$\mathrm{g} = 3\{-Aa - Baz - 2Caz^2 + Cbz^3\},$$

$$\mathrm{b} = 3\{Aa + (7Ba - Ab)\,z - 2(Bb - Ca)\,z^2 - 2Bcz^3\},$$

$$\mathrm{p} = Ab - Ba + 3Bbz + 3Cbz^2 + (Cc - Db)\,z^3,$$

$$\mathrm{f} = 3\{-Ba - 2(Bb - Ca)\,z + 2(Bc - Cb)\,z^2 - Ccz^3\},$$

$$\mathrm{c} = 3\{-2Ca + 2(Bc - Cb)\,z + (7Cc - Db)\,z^2 + Dcz^3\},$$

$$\mathrm{q} = 3\{Bb - 2Bcz - Ccz^2 - Dcz^3\},$$

$$\mathrm{r} = 3(Cb - Ccz + Dcz^2),$$

$$\mathrm{d} = Db - Cc - 3Dcz.$$

This result may be verified as follows:
Writing the equation

$$XYZ = P^3,$$

where

$$P = A + B(x + y + z) + C(yz + zx + xy) + Dxyz;$$

and taking the logarithmic differential

$$X^{-1}\frac{dX}{dx}dx + Y^{-1}\frac{dY}{dy}dy + Z^{-1}\frac{dZ}{dz}dz = 3P^{-1}\left(\frac{dP}{dx}dx + \frac{dP}{dy}dy + \frac{dP}{dz}dz\right);$$

or

$$\left(\tfrac{1}{3}X^{-1}\frac{dX}{dx} - P^{-1}\frac{dP}{dx}\right)dx + \left(\tfrac{1}{3}Y^{-1}\frac{dY}{dy} - P^{-1}\frac{dP}{dy}\right)dy + \left(\tfrac{1}{3}Z^{-1}\frac{dZ}{dz} - P^{-1}\frac{dP}{dz}\right)dz = 0.$$

Comparing this with the original equation

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0,$$

we ought to have

$$Y^{\frac{1}{3}}\left(\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)=X^{\frac{1}{3}}\left(\tfrac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right),$$

or

$$Y\left(\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)^3-X\left(\tfrac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)^3=0.$$

It is to be shewn that

$$XYZ-P^3$$

is a factor of the left-hand side of this equation, which is therefore true.

We have

$$\tfrac{1}{3}\frac{dX}{dx}=B+2Cx+Dx^2,$$

$$\frac{dP}{dx}=B+C(y+z)+Dyz;$$

and if

$$B^2-AC=a,\quad AD-BC=b,\quad C^2-BD=c,$$

$$\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}=a(y+z)-byz+2(cyz-a)x+\{b-c(y+z)\}x^2.$$

Putting, for convenience, the coefficient of x^m in

$$\left\{\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right\}^3,$$

equal to $\quad A_m+B_my+C_my^2+D_my^3,$

we find on multiplying out

$$Y\left(\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)^3-X\left(\tfrac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)^3$$

$$\begin{aligned}
=\ & (AB_0+3BA_0-AA_1)(y-x)\\
&+(AC_0+3BB_0+3CA_0-AA_2)(y^2-x^2)\\
&+(AD_0+3BC_0+3CB_0+DA_0-AA_3)(y^3-x^3)\\
&+(AC_1+3BB_1+3CA_1-AB_2-3BA_2)xy(y-x)\\
&+(3BD_0+3CC_0+DB_0-AA_4)(y^4-x^4)
\end{aligned}$$

$$+ (AD_1 + 3BC_1 + 3CB_1 + DA_1 - AB_3 - 3BA_3)\, xy\,(y^2 - x^2)$$
$$+ (3CD_0 + DC_0 - AA_5)\,(y^5 - x^5)$$
$$+ (3BD_1 + 3CC_1 + DB_1 - AB_4 - 3BA_4)\, xy\,(y^3 - x^3)$$
$$+ (AD_2 + 3BC_2 + 3CB_2 + DA_2 - AC_3 - 3BB_3 - 3CA_3)\, x^2y^2\,(y - x)$$
$$+ (DD_0 - AA_6)\,(y^6 - x^6)$$
$$+ (3CD_1 + DC_1 - AB_5 - 3BA_5)\, xy\,(y^4 - x^4)$$
$$+ (3BD_2 + 3CC_2 + DB_2 - AC_4 - 3BB_4 - 3CA_4)\, x^2y^2\,(y^2 - x^2)$$
$$+ (DD_1 - AB_6 - 3BA_6)\, xy\,(y^5 - x^5)$$
$$+ (3CD_2 + DC_2 - AC_5 - 3BB_5 - 3CA_5)\, x^2y^2\,(y^3 - x^3)$$
$$+ (3BD_3 + 3CC_3 + DB_3 - AD_4 - 3BC_4 - 3CB_4 - DA_4)\, x^3y^3\,(y - x)$$
$$+ (DD_2 - AC_6 - 3BB_6 - 3CA_6)\, x^2y^2\,(y^4 - x^4)$$
$$+ (3CD_3 + DC_3 - AD_5 - 3BC_5 - 3CB_5 - DA_5)\, x^3y^3\,(y^3 - x^3)$$
$$+ (DD_3 - AD_6 - 3BC_6 - 3CB_6 - DA_6)\, x^3y^3\,(y^3 - x^3)$$
$$+ (3CD_4 + DC_4 - 3BD_5 - 3CC_5 - DB_5)\, x^4y^4\,(y - x)$$
$$+ (DD_4 - 3BD_6 - 3CC_6 - DB_6)\, x^4y^4\,(y^2 - x^2)$$
$$+ (DD_5 - 3CD_6 - DC_6)\, x^5y^5\,(y - x)$$

which is divisible by $y - x$.

Whence, casting out this factor and rearranging the terms, the expression becomes

$$AB_0 + 3BA_0 - AA_1$$
$$+ (x + y)\,(AC_0 + 3BB_0 + 3CA_0 - AA_2)$$
$$+ (x^2 + y^2)\,(AD_0 + 3BC_0 + 3CB_0 + DA_0 - AA_3) + xy\,(AD_0 + 3BC_0$$
$$+ 3CB_0 + DA_0 - AA_3 + AC_1 + 3BB_1 + 3CA_1 - AB_2 - 3BA_2)$$
$$+ (x^3 + y^3)\,(3BD_0 + 3CC_0 + DB_0 - AA_4) + xy\,(x + y)\,(3BD_0 + 3CC_0$$
$$+ DB_0 - AA_4 + AD_1 + 3BC_1 + 3CB_1 + DA_1 - AB_3 - 3BA_3)$$
$$+ (x^4 + y^4)\,(3CD_0 + DC_0 - AA_5) + xy\,(x^2 + y^2)\,(3CD_0 + DC_0 - AA_5$$
$$+ 3BD_1 + 3CC_1 + DB_1 - AB_4 - 3BA_4) + x^2y^2\,(3CD_0 + DC_0$$
$$- AA_5 + 3BD_1 + 3CC_1 + DB_1 - AB_4 - 3BA_4 + AD_2 + 3BC_2$$
$$+ 3CB_2 + DA_2 - AC_3 - 3BB_3 - 3CA_3)$$

$$+(x^5+y^5)(DD_0-AA_6)+xy(x^3+y^3)(DD_0-AA_6+3CD_1+DC_1$$
$$-AB_5-3BA_5)+x^2y^2(x+y)(DD_0-AA_6+3CD_1+DC_1$$
$$-AB_5-3BA_5+3BD_2+3CC_2+DB_2-AC_4-3BB_4-3CA_4)$$
$$+xy(x^4+y^4)(DD_1-AB_6-3BA_6)+x^2y^2(x^2+y^2)(DD_1-AB_6-3BA_6$$
$$+3CD_2+DC_2-AC_5-3BB_5-3CA_5)+x^3y^3(DD_1-AB_6$$
$$-3BA_6+3CD_2+DC_2-AC_5-3BB_5-3CA_5+3BD_3+3CC_3$$
$$+DB_3-AD_4-3BC_4-3CB_4-DA_4)$$
$$+x^2y^2(x^3+y^3)(DD_2-AC_6-3BB_6-3CA_6)+x^3y^3(x+y)(DD_2-AC_6$$
$$-3BB_6-3CA^6+3CD_3+DC_3-AD_5-3BC_5-3CB_5-DA_5)$$
$$+x^3y^3(x^2+y^2)(DD_3-AD_6-3BC_6-3CB_6-DA_6)+x^4y^4(DD_3-AD_6$$
$$-3BC_6-3CB_6-DA_6+3CD_4+DC_4-3BD_5-3CC_5-DB_5)$$
$$+x^4y^4(x+y)(DD_4-3BD_6-3CC_6-DB_6)$$
$$+x^5y^5(DD_5-3CD_6-DC_6).$$

To calculate these coefficients, we require the values of the letters with suffixes from the development of

$$[a(y+z)-byz+2(cyz-a)x+\{b-c(y+z)\}x^2]^3.$$

Thence we find

$$A_0 = a^3z^3,$$
$$B_0 = 3a^2(az^2-bz^3),$$
$$C_0 = 3a(a^2z-2abz^2+b^2z^3),$$
$$D_0 = a^3-3a^2bz+3ab^2z^2-b^3z^3,$$
$$A_1 = -6a^3z^2,$$
$$B_1 = 6a^2(-2az+2bz^2+cz^3),$$
$$C_1 = 6a\{-a^2+2abz+(2ac-b^2)z^2-2bcz^3\},$$
$$D_1 = bc(a^2z-2abz^2+b^2z^3),$$
$$A_2 = 3a^2(4az+bz^2-cz^3),$$
$$B_2 = 3a\{4a^2-2abz-(11ac+2b^2)z^2+2bcz^3\},$$
$$C_2 = 3\{a^2b-a(11ac+2b^2)z+b(12ac+b^2)z^2+c(4ac-b^2)z^3\},$$
$$D_2 = 3c\{-a^2+2abz+(4ac-b^2)z^2-4bcz^3\},$$

$A_3 = 4a^2(-2a - 3bz + 3cz^2)$,

$B_3 = 12a\{-ab + (4ac + b^2)z - c^2z^3\}$,

$C_3 = 12c\{a^2(4ac + b^2)z^2 + bcz^3\}$,

$D_3 = 4c^2(-3az + 3bz^2 + 2cz^3)$,

$A_4 = 3a\{4ab - (4ac - b^2)z - 2bcz^2 + c^2z^3\}$,

$B_4 = 3\{-a(4ac - b^2) - b(12ac + b^2)z + c(11ac + 2b^2)z^2 - bc^2z^3\}$,

$C_4 = 3c\{-2ab + (11ac + 2b^2)z + 2bcz^2 - 4c^2z^3\}$,

$D_4 = 3c^2(a - bz - 4cz^2)$,

$A_5 = -6a(b^2 - 2bcz + c^2z^2)$,

$B_5 = 6c\{2ab - (2ac - b^2)z - 2bcz^2 + c^2z^3\}$,

$C_5 = -bc^2(a + 2bz - 2cz^2)$,

$D_5 = 6c^3z$,

$A_6 = b^3 - 3b^2cz + 3bc^2z^2 - c^3z^3$,

$B_6 = 3c(-b^2 + 2bcz - c^2z^2)$,

$C_6 = 3c^2(b - cz)$,

$D_6 = -c^3$.

From these values we find that in the expression we are considering,

$$\text{absolute term} = 3a^2\{3Aaz^2 - (Ab - Ba)z^3\}$$
$$= -3a^2\mathrm{a};$$

where a is the first of the ten coefficients of the general equation given *ante*.

Coefficient of $x + y$

$= 3a\{-3Aa^2z + 3a(Ba - Ab)z^2 + (Ab^2 + Aac - 3Bab + Ca^2)z^3\}$

$= 3a\{-3Aabz^2 + b(Ab - Ba)z^3 - 3Aa^2z + 3Ba^2z^2 - 3Babz^3\}$

(since $Aa + Bb + Ca = 0$)

$= 3a(b\mathrm{a} - a\mathrm{h}) = 3ab\mathrm{a} - 3a^2\mathrm{h}$;

of $x^2 + y^2$,

$= 9Aa^3 + 9a^2(Ab + Ba)z + 3a(Ab^2 - 6Bab + 3Ca^2 - 4Aac)z^2$
$+ (Da^3 - 9Ca^2b + 9Bab^2 - Ab^3)z^3$

$= -3a^2\mathrm{g} + 3ab\mathrm{h} - (b^2 - ac)\mathrm{a}$;

as can be readily verified,

of xy,

$$= -9Aa^3 + 9a^2(3Ab - 7Ba)\,z + 3a\,\{A(11ac + b^2) + 3Bab - 3Ca^2\}z^2$$
$$+ \{-Ab(b^2 + 18ac) + 9Ba\,(b^2 + 3ac) - 9Ca^2b + Da^3\}\,z^3$$
$$= -3a^2\mathrm{b} + 6ab\mathrm{h} - (b^2 + 8ac)\,\mathrm{a};$$

of $x^3 + y^3$,

$$= 3a^2\,(-4Ab + Ba) + 3a\,\{A\,(4ac - b^2) - 3Bab + 3Ca^2\}\,z$$
$$+ 3a\,(2Abc + 3Bb^2 - 6Cab + Da^2)\,z^2$$
$$+ 3\,(-Aac^2 - Bb^3 + 3Cab^2 - Da^2b)\,z^3,$$
$$= -3a^2\mathrm{p} + 3ab\mathrm{g} - (b^2 - ac)\,\mathrm{h};$$

of $xy\,(x + y)$,

$$= 9Ba^3 + 3a\,\{-5A\,(2ac + b^2) + 21Bab - 9Ca^2\}\,z$$
$$+ 3a\,(-2Abc - 3Bb^2 + 6Cab - Da^2)\,z^2$$
$$+ 3\,\{Ac\,(3ac + 2b^2) - Bb\,(b^2 + 12ac) + 3Ca\,(b^2 + 2ac) - Da^2b\}\,z^3$$
$$= -3a^2\mathrm{f} + 3ab\mathrm{g} + 3ab\mathrm{b} + 3bc\mathrm{a} - (b^2 - ac)\,\mathrm{h} - (b^2 + 8ac)\,\mathrm{h};$$

of $x^4 + y^4$,

$$= 3a\,(2Ab^2 + Ca^2) + 3a\,(-4Abc - 3Cab + Da^2)\,z$$
$$+ 3a\,(2Ac^2 + 3Cb^2 - 2Dab)\,z^2 + 3b^2\,(Da - Cb)\,z^3$$
$$= 3ab\mathrm{p} - (b^2 - ac)\,\mathrm{g};$$

of $xy\,(x^2 + y^2)$,

$$= 3a\,\{A\,(b^2 + 4ac) - 12Bab - 5Ca^2\}$$
$$+ 3\ \{Ab\,(b^2 + 8ac) + 3Ba\,(6ac - b^2) + 9Ca^2b - 3Da^3\}\,z$$
$$+ 3\ \{-Ac\,(2b^2 + 9ac) - 6Babc + 3Ca\,(4ac - b^2) + 2Da^2b\}\,z^2$$
$$+ 3\ \{Abc^2 + 3Bc\,(2b^2 - ac) - Cb\,(12ac + b^2) + Da\,(2ac + b^2)\}\,z^3$$
$$= -3a^2\mathrm{q} + 3ab\mathrm{p} + 3ab\mathrm{f} - (b^2 + 8ac)\,\mathrm{g} - (b^2 - ac)\,\mathrm{b} + 3bc\mathrm{h};$$

of x^2y^2,

$$= 3a\,\{A\,(b^2 - ac) + 3Bab + 15Ca^2\}$$
$$+ 3\ \{Ab\,(b^2 + 10ac) - 21Ba\,(b^2 + 3ac) + 15Ca^2b + Da^3\}\,z$$
$$+ 3\ \{Ac\,(11ac + b^2) + 3Bb\,(b^2 + 10ac) - 3Ca\,(3b^2 + 11ac) + 3Da^2b\}\,z^2$$
$$+ 3\ \{-7Abc^2 + 3Bc\,(b^2 + 7ac) - Cb\,(b^2 + 6ac) + Da\,\{b^2 + ac)\}\,z^3$$
$$= -3a^2\mathrm{c} + 6ab\mathrm{f} - (b^2 + 8ac)\,\mathrm{b} - 2\,(b^2 - ac)\,\mathrm{g} + 6bc\mathrm{h} - 3c^2\mathrm{a};$$

of $x^5 + y^5$,

$$= (-Ab^3 + Da^3) + 3b(Abc - Da^2)z + 3b(-Ac^2 + Dab)z^2 + (Ac^3 - Db^3)z^3$$

$$= -(b^2 - ac)\,\mathrm{p};$$

of $xy(x^3 + y^3)$,

$$= -Ab(b^2 + 12ac) + 18Bab^2 - 5Da^3 + 3\{-Ac(b^2 - 4ac) - 12Babc + 6Ca^2c + 3Da^2b\}z + 3\{3Abc^2 + 6Bac^2 - 12Cabc + Da(4ac - b^2)\}z^2 + \{-5Ac^3 + 18Cb^2c - Db(b^2 + 12ac)\}z^3$$

$$= 3abq - (b^2 + 8ac)\,\mathrm{p} - (b^2 - ac)\,\mathrm{f} + 3bcg;$$

of $x^2y^2(x + y)$,

$$= -Ab(b^2 + 6ac) + 9Ba(b^2 + 3ac) - 27Ca^2b + 7Da^3 + 3\{-Ac(3b^2 + 7ac) + 3Bb(b^2 + 10ac) - 3Ca(3b^2 + 5ac) + Da^2b\}z + 3\{Abc^2 - 3Bc(3b^2 + 5ac) + 3Cb(b^2 + 10ac) - Da(3b^2 + 7ac)\}z^2 + \{7Ac^3 - 27Bbc^2 + 9Cc(3ac + b^2) - Db(b^2 + 6ac)\}z^3,$$

$$= -3a^2r + 3abc + 3abq - (b^2 + 8ac)\,\mathrm{f} - (b^2 - ac)\,\mathrm{p} - (b^2 - ac)\,\mathrm{f} + 3bcg + 3bcb - 3c^2h;$$

of $xy(x^4 + y^4)$,

$$= 3b^2(Ac - Bb) + 3c\{-2Abc + 3Bb^2 + 2Da^2\}z + 3c(Ac^2 - 3Bbc - 4Dab)z^2 + 3c(Bc^2 + 2Db^2)z^3$$

$$= -(b^2 - ac)\,\mathrm{q} + 3bcp;$$

of $x^2y^2(x^2 + y^2)$,

$$= 3\{Ac(b^2 + 2ac) - Bb(b^2 + 12ac) + 3Ca(2b^2 - ac) + Da^2b\} + 3\{2Abc^2 + 3Bc(4ac - b^2) - 6Cabc - Da(2b^2 + 9ac)\}z + 3\{-3Ac^3 + 9Bbc^2 + 3Cc(6ac - b^2) + Db(b^2 + 8ac)\}z^2 + 3c\{-5Bc^2 - 12Cbc + D(4ac + b^2)\}z^3$$

$$= 3abr - (b^2 + 8ac)\,\mathrm{q} - (b^2 - ac)\,\mathrm{c} + 3bcp + 3bcf - 3c^2g;$$

of x^3y^3,

$$= \quad 3 \ \{Ac\,(b^2+ac) - Bb\,(b^2+6ac) + 3\,Ca\,(b^2+7ac) - 7\,Da^2b\}$$
$$+ 3 \ \{3Abc^2 - 3Bc\,(3b^2+11ac) + 3\,Cb(b^2+10ac) + Da(b^2+11ac)\}z$$
$$+ 3 \ \{Ac^3 + 15Bbc^2 - 21\,Cc\,(b^2+3ac) + Db\,(b^2+10ac)\}z^2$$
$$+ 3c\,\{15Bc^2 + 3\,Cbc + D\,(b^2-ac)\}z^3$$
$$= -\,3a^2\mathrm{d} + 6ab\mathrm{r} - (b^2+8ac)\,\mathrm{c} - 2\,(b^2-ac)\,\mathrm{q} + 6bc\mathrm{f} - 3c^2b\,;$$

of $x^2y^2\,(x^3+y^3)$,

$$= \quad 3(-Abc^2 + 3Bb^2c - Cb^3 - Da^2c) + 3(Ac^3 - 6Bbc^2 + 3\,Cb^2c + 2Dabc)z$$
$$+ 3\,\{3Bc^3 - 3\,Cbc^2 + Dc\,(4ac-b^2)\}\,z^2 + 3c^2\,(Cc - 4Db)\,z^3,$$
$$= - \quad (b^2-ac)\,\mathrm{r} + 3bc\mathrm{q} - 3c^2\mathrm{p}\,;$$

of $x^3y^3\,(x+y)$,

$$= \quad 3\,\{-Abc^2 + 3Bc\,(b^2+2ac) - Cb\,(b^2+12ac) + Da\,(2b^2+3ac)\}$$
$$+ 3\,(-Ac^3 + 6Bbc^2 - 3\,Cb^2c - 2Dabc)\,z$$
$$+ 3\,\{-9Bc^3 + 21\,Cbc^2 - 5Dc\,(b^2+2ac)\}\,z^2 + 9\,Cc^3z^3$$
$$= \quad 3\,ab\mathrm{d} - (b^2+8ac)\,\mathrm{r} - (b^2-ac)\,\mathrm{r} + 3bc\mathrm{c} + 3bc\mathrm{q} - 3c\mathrm{f}^2\,;$$

of $x^3y^3\,(x^2+y^2)$,

$$= \quad Ac^3 - 9Bbc^2 + 9\,Cb^2c - Db^3$$
$$+ 3c\,\{3Bc^2 - 6\,Cbc + D\,(b^2-4ac)\}\,z + 9c^2\,(Cc+Db)z^2 + 9Dc^3z^3,$$
$$= - \quad (b^2-ac)\,\mathrm{d} + 3bc\mathrm{r} - 3c^2\mathrm{q}\,;$$

of x^4y^4,

$$= \quad Ac^3 - 9Bbc^2 + 9\,Cc\,(b^2+3ac) - Db\,(b^2+18ac)$$
$$+ 3c\,\{-3Bc^2 + 3\,Cbc + D\,(b^2+11ac)\}\,z$$
$$+ 9c^2\,(-7\,Cc + 3Db)\,z^2 - 9Dc^3z^3,$$
$$= - \,(b^2+8ac)\,\mathrm{d} + 6bc\mathrm{r} - 3c^2C\,;$$

of $x^4y^4\,(x+y)$

$$= \quad 3c\,\{Bc^2 - 3\,Cbc + D\,(b^2+ac) + 3c\,(Cc - Db)\,z - 3Dc^2z^2\}$$
$$= \quad 3bc\mathrm{d} - 3c^2\mathrm{r}\,;$$

of x^5y^5

$$= \quad 3c^2\,(Cc - Db) + 9Dc^3z,$$
$$= - \,3c^2\mathrm{d}.$$

The symmetry of the coefficient is very noticeable, viz. in those terms similarly situated in regard to the two ends of the expression.

Substituting these values in the expression, we find it to be equal to

$$\left\{\begin{matrix} -3a^2+3ab\,(x+y)-(b^2+8ac)\,xy-(b^2-ac)\,(x^2+y^2) \\ +3bcxy\,(x+y)-3c^2x^2y^2 \end{matrix}\right\}$$

multiplied by

$$\begin{aligned} &\mathrm{a}+\mathrm{h}\,(x+y)+\mathrm{g}\,(x^2+y^2)+\mathrm{b}xy+\mathrm{p}\,(x^3+y^3) \\ &+\mathrm{f}xy\,(x+y)+\mathrm{c}x^2y^2+\mathrm{q}xy\,(x^2+y^2) \\ &+\mathrm{r}x^2y^2\,(x+y)+\mathrm{d}x^3y^3; \end{aligned}$$

the latter factor being

$$\begin{aligned} &\mathrm{a}+\mathrm{h}y+\mathrm{g}y^2+\mathrm{p}y^3 \\ &+x\ (\mathrm{h}+\mathrm{b}y+\mathrm{f}y^2+\mathrm{q}y^3) \\ &+x^2\,(\mathrm{g}+\mathrm{f}y+\mathrm{c}y^2+\mathrm{r}y^3) \\ &+x^3\,(\mathrm{p}+\mathrm{q}y+\mathrm{r}y^2+\mathrm{d}y^3); \end{aligned}$$

so that finally we have shewn that

$$Y\left(\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)^3-X\left(\tfrac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)^3$$

$$\begin{aligned} =(XYZ-P^3)\,(x-y)\,\{&3a^2-3ab\,(x+y)+(b^2+8ac)\,xy \\ &+(b^2-ac)\,(x^2+y^2)-3bcxy\,(x+y)+3c^2x^2y^2\}, \end{aligned}$$

where $a=B^2-AC$; $b=AD-BC$; $c=C^2-BD$;

and $P=A+B\,(x+y+z)+C\,(yz+zx+xy)+Dxyz$,

which completes the verification.

In the case

$$(1-x^3)^{-\frac{2}{3}}dx+(1-y^3)^{-\frac{2}{3}}dy=0,$$

we have at once for the general integral

$$(1-x^3)\,(1-y^3)\,(1-z^3)=(1-xyz)^3,$$

$z=\infty$, $x^3+y^3=1$;

$z=1$, $xy=1$;

$z=0$, $x^3+y^3-x^3y^3=0$.

The envelope of the series of curves represented by

$$(1-x^3)(1-y^3)(1-z^3)=(1-xyz)^3;$$

z, being a variable parameter, is found at once by the method given in Salmon's *Higher Plane Curves*.

It is found on reduction to consist of the three lines

$$x-y=0,$$
$$x-1=0,$$
$$y-1=0.$$

Writing X, Y, Z in the forms

$$X=(x-\alpha)(x-\beta)(x-\gamma),$$

with similar expressions for Y and Z, the general integral may be written

$$27(x-\alpha)(x-\beta)(x-\gamma)(y-\alpha)(y-\beta)(y-\gamma)(z-\alpha)(z-\beta)(z-\gamma)$$
$$=\{-3\alpha\beta\gamma+(\beta\gamma+\gamma\alpha+\alpha\beta)(x+y+z)$$
$$-(\alpha+\beta+\gamma)(yz+zx+xy)+3xyz\}^3,$$

or otherwise

$$27(\alpha-x)(\alpha-y)(\alpha-z)(\beta-x)(\beta-y)(\beta-z)(\gamma-x)(\gamma-y)(\gamma-z)$$
$$=\{-3xyz+(yz+zx+xy)(\alpha+\beta+\gamma)$$
$$-(x+y+z)(\beta\gamma+\gamma\alpha+\alpha\beta)+3\alpha\beta\gamma\}^3.$$

The symmetry evident here shows that if

$$\frac{dx}{\{(x-\alpha)(x-\beta)(x-\gamma)\}^{\frac{2}{3}}}+\frac{dy}{\{(y-\alpha)(y-\beta)(y-\gamma)\}^{\frac{2}{3}}}$$
$$+\frac{dz}{\{(z-\alpha)(z-\beta)(z-\gamma)\}^{\frac{2}{3}}}=0,$$

when α, β and γ are constant, then also

$$\frac{d\alpha}{\{(\alpha-x)(\alpha-y)(\alpha-z)\}^{\frac{2}{3}}}+\frac{d\beta}{\{(\beta-x)(\beta-y)(\beta-z)\}^{\frac{2}{3}}}$$
$$+\frac{d\gamma}{\{(\gamma-x)(\gamma-y)(\gamma-z)\}^{\frac{2}{3}}}=0,$$

when x, y and z are constant.

The variables in the general integral may be each expressed in terms of a single variable t.

For since

$$x_1 = b + b\sqrt{3}\,\frac{1+\operatorname{cn} u}{1-\operatorname{cn} u},$$

$$y_1 = b + b\sqrt{3}\,\frac{1+\operatorname{cn} v}{1-\operatorname{cn} v},$$

$$z_1 = b + b\sqrt{3}\,\frac{1+\operatorname{cn} w}{1-\operatorname{cn} w};$$

and then

$$u + v + w = 0;$$

we may write

$$u = -\tfrac{1}{2}w + t,$$

$$v = -\tfrac{1}{2}w - t;$$

and we find easily

$$x_1^3 - b^3 = 12\sqrt{3}b^3\,\frac{\operatorname{sn}^2(t-\frac{1}{2}w)\operatorname{dn}^2(t-\frac{1}{2}w)}{\{1-\operatorname{cn}(t-\frac{1}{2}w)\}^4},$$

$$y_1^3 - b^3 = 12\sqrt{3}b^3\,\frac{\operatorname{sn}^2(t+\frac{1}{2}w)\operatorname{dn}^2(t+\frac{1}{2}w)}{\{1-\operatorname{cn}(t+\frac{1}{2}w)\}^4},$$

whence

$$\frac{(\alpha-p)x - p\alpha + q}{(x-\alpha)\sqrt{(\alpha^2-2p\alpha+q)}} = \sqrt{(12\sqrt{3}b^3)}\,\frac{\operatorname{sn}(t-\frac{1}{2}w)\operatorname{dn}(t-\frac{1}{2}w)}{\{1-\operatorname{cn}(t-\frac{1}{2}w)\}^2},$$

$$\frac{(\alpha-p)y - p\alpha + q}{(y-\alpha)\sqrt{(\alpha^2-2p\alpha+q)}} = \sqrt{(12\sqrt{3}b^3)}\,\frac{\operatorname{sn}(t+\frac{1}{2}w)\operatorname{dn}(t+\frac{1}{2}w)}{\{1-\operatorname{cn}(t+\frac{1}{2}w)\}^2},$$

and therefore

$$x = \frac{\alpha\sqrt{\{12\sqrt{3}(q-p^2)\}}\operatorname{sn}(t-\frac{1}{2}w)\operatorname{dn}(t-\frac{1}{2}w) - (p\alpha-q)\{1-\operatorname{cn}(t-\frac{1}{2}w)\}^2}{\sqrt{\{12\sqrt{3}(q-p^2)\}}\operatorname{sn}(t-\frac{1}{2}w)\operatorname{dn}(t-\frac{1}{2}w) - (\alpha-p)\{1-\operatorname{cn}(t-\frac{1}{2}w)\}^2},$$

$$y = \frac{\alpha\sqrt{\{12\sqrt{3}(q-p^2)\}}\operatorname{sn}(t+\frac{1}{2}w)\operatorname{dn}(t+\frac{1}{2}w) - (p\alpha-q)\{1-\operatorname{cn}(t+\frac{1}{2}w)\}^2}{\sqrt{\{12\sqrt{3}(q-p^2)\}}\operatorname{sn}(t+\frac{1}{2}w)\operatorname{dn}(t+\frac{1}{2}w) - (\alpha-p)\{1-\operatorname{cn}(t+\frac{1}{2}w)\}^2},$$

w being the arbitrary constant, and

$$\operatorname{cn} w = \frac{Z^{\frac{1}{3}} - b(\sqrt{3}+1)(z-\alpha)}{Z^{\frac{1}{3}} + b(\sqrt{3}-1)(z-\alpha)}.$$

The general integral might be obtained by the elimination of t between these two equations.

§ 2. The rational algebraical integral of the differential equation

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + V^{-\frac{2}{3}}dv + W^{-\frac{2}{3}}dw = 0,$$

where

$$X = A + 3Bx + 3Cx^2 + Dx^3,$$
$$Y = A + 3By + 3Cy^2 + Dy^3,$$
$$V = A + 3Bv + 3Cv^2 + Dv^3,$$
$$W = A + 3Bw + 3Cw^2 + Dw^3,$$

may be found as a particular solution, and as the general integral of

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + V^{-\frac{2}{3}}dy = 0,$$

w being the arbitrary constant.

For with the previous notation, if we put

$$v_1 = b + b\sqrt{3}\,\frac{1 + \operatorname{cn} u'}{1 - \operatorname{cn} u'},$$

$$w_1 = b + b\sqrt{3}\,\frac{1 + \operatorname{cn} v'}{1 - \operatorname{cn} v'},$$

$$z_1 = b + b\sqrt{3}\,\frac{1 + \operatorname{cn} a}{1 - \operatorname{cn} a},$$

where

$$\int_{v_1}^{\infty} \frac{dv_1}{V_1} + \int_{w_1}^{\infty} \frac{dw_1}{W_1} - \int_{z_1}^{\infty} \frac{dZ_1}{Z_1} = 0.$$

Then

$$u' + v' = a,$$

and when

$v_1 = w_1$, then z_1 is finite;

$w_1 = z_1$, „ z_1 is infinite;

$z_1 = v_1$, „ w_1 is infinite;

so that we can write Euler's integral in the forms

$$v_1 + w_1 + z_1 = \left\{\frac{V_1 - W_1}{v_1 - w_1}\right\}^2,$$
$$= \left\{\frac{W_1 + Z_1}{w_1 - z_1}\right\}^2,$$
$$= \left\{\frac{Z_1 + V_1}{z_1 - v_1}\right\}^2;$$

where, as before, $V_1 = \sqrt{(v_1^3 - b^3)}$, &c.,

leading to

$$3v_1w_1z_1 = 3b^3 - W_1Z_1 - Z_1V_1 + V_1W_1 \ldots\ldots\ldots (1),$$

and, since then,

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0,$$
$$V^{-\frac{2}{3}}dv + W^{-\frac{2}{3}}dw - Z^{-\frac{2}{3}}dz = 0;$$

therefore

$$X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + V^{-\frac{2}{3}}dv + W^{-\frac{2}{3}}dw = 0,$$

and the corresponding relation between

$$x,\quad y,\quad v,\quad w$$

will be the result of eliminating z between the two equations

$$3x_1y_1z_1 = 3b^3 + Y_1z_1 + Z_1X_1 + XY_1,$$
$$3v_1w_1z_1 = 3b^3 - W_1z_1 - Z_1V_1 + V_1W_1.$$

These equations when developed are respectively

$$XYZ = (R + Sz)^3,$$
$$VWZ = (P + Qz)^3,$$

where
$$R = A + B(x + y) + Cxy,$$
$$S = B + C(x + y) + Dxy,$$

$$3(\alpha^2 - 2p\alpha + q)P = (v + w - \alpha)(3\alpha^2q - 4\alpha^2p^2 + 2pq\alpha - q^2) + vw(2pq - 5\alpha q + 4\alpha^2p - \alpha^3),$$

$$3(\alpha^2 - 2p\alpha + q)Q = 5\alpha^2q - 4pq\alpha - 2p\alpha^3 + q^2 + (vw - \alpha v - \alpha w)(3q + 2\alpha p - 4p^2 - \alpha^2),$$

whence
$$\frac{R + Sz}{P + Qz} = \left(\frac{XY}{VW}\right)^{\frac{1}{3}},$$

therefore
$$z = \frac{PX^{\frac{1}{3}}Y^{\frac{1}{3}} - RV^{\frac{1}{3}}W^{\frac{1}{3}}}{SV^{\frac{1}{3}}W^{\frac{1}{3}} - QX^{\frac{1}{3}}Y^{\frac{1}{3}}};$$

or, substituting this value in either of the two equations, the first suppose, multiplying up by the cube of

$$SV^{\frac{1}{3}}W^{\frac{1}{3}} - QX^{\frac{1}{3}}Y^{\frac{1}{3}},$$

and putting $XY = TVW$,
we have

$$AVW(S - QT^{\frac{1}{3}}) + 3BVW(PT^{\frac{1}{3}} - R)(S - QT^{\frac{1}{3}})^2$$
$$+ 3CVW(PT^{\frac{1}{3}} - R)(S - QT^{\frac{1}{3}}) + DVW(PT^{\frac{1}{3}} - R) = (PS - QR)^3.$$

Arranging according to powers of $T^{\frac{1}{3}}$, we have

$$A_1 T^{\frac{2}{3}} + B_1 T^{\frac{1}{3}} = C_1,$$

where

$$A_1 = 3VW\{AQ^2S - BQ(2PS + QR) + CP(PS + 2QR) - DP^2R\},$$
$$B_1 = 3VW\{-AQS^2 + BS(PS + QR) - CR(2PS + QR) + DPR^2\},$$
$$C_1 = (PS - QR)^3 + XY(AQ^3 - 3BPQ^2 + 3CP^2Q - DP^3)$$
$$- VW(AS^3 - 3BRS^2 + 3CR^2S - DR^3),$$

or cubing

$$A_1^3 T^2 + 3A_1 B_1 (A_1 T^{\frac{2}{3}} + B_1 T^{\frac{1}{3}}) + B_1^3 T = C_1^3,$$

or

$$A_1^3 T^2 + B_1^3 T + 3A_1 B_1 C_1 = C_1^3.$$

Therefore putting $\frac{XY}{VW}$ for T,

$$(3A_1 B_1 C_1 + B_1^3) XYVW = C_1^3 V^2 W^2 - A_1^3 X^2 Y^2,$$

and putting

$$A_1 = 3a_1 VW,$$
$$B_1 = 3b_1 VW;$$

$$27(a_1 b_1 C_1 + b_1^3 VW) XYVW = C_1^3 - 27a_1^3 VWX^2 Y^2;$$

therefore

$$27(a_1^3 XY + b_1^3 VW + a_1 b_1 C_1) XYVW = C_1^3,$$

which is the rational algebraical integral of

$$X^{-\frac{2}{3}} dx + Y^{-\frac{2}{3}} dy + Z^{-\frac{2}{3}} dz = 0,$$

w being the arbitrary constant, and a_1, b_1, C_1 having the values given above.

As a particular case of the above general result, consider the dfferential equation

$$(1 - x^3)^{-\frac{2}{3}} dx + (1 - y^3)^{-\frac{2}{3}} dy + (1 - v^3)^{-\frac{2}{3}} dv = 0,$$

and take w the arbitrary constant, such that

$$X^{-\frac{2}{3}} dx + Y^{-\frac{2}{3}} dy + V^{-\frac{2}{3}} dv + W^{-\frac{2}{3}} dw = 0.$$

The two equations, between which we eliminate z, are then

$$XYZ = (1 - xyz)^3,$$

and

$$VWZ = (vw - z)^3;$$

so that

$$R = 1,$$

$$S = -xy,$$

$$P = vw,$$

$$Q = -1,$$

$$A = 1, \quad B = 0, \quad C = 0, \quad D = -1,$$

$$a_1 = v^2w^2 - xy, \quad b_1 = x^2y^2 - vw,$$

$$C_1 = (1 - xyvw)^3 + (v^3w^3 - 1)\,XY + (x^3y^3 - 1)\,VW;$$

therefore

$$a_1^3 XY + b_1^3 VW + a_1 b_1 C_1$$

$$= a_1\{a_1^2 + b_1(v^3w^3 - 1)\}\,XY + b_1\{b_1^2 + a_1(x^3y^3 - 1)\}\,VW + a_1 b_1 (1 - xyvw)^3,$$

$$= vw\,(v^2w^2 - xy)(1 - xyvw)^2\,XY + xy\,(x^2y^2 - vw)(1 - xyvw)^2\,VW + (v^2w^2 - xy)\,(x^2y^2 - vw)\,(1 - xyvw)^3,$$

$$= (1 - xyvw)^2\{vw\,(v^2w^2 - xy)\,XY + xy\,(x^2y^2 - vw)\,VW + (v^2w^2 - xy)\,(x^2y^2 - vw)\,(1 - xyvw)\};$$

therefore substituting

$$27(1 - xyvw)^2\{(v^2w^2 - xy)(x^2y^2 - vw)(1 - xyvw) + vw(v^2w^2 - xy)XY + xy\,(x^2y^2 - vw)\,VW\}\,XYVW$$

$$= \{(1 - xyvw)^3 - (1 - v^3w^3)\,XY - (1 - x^3y^3)\,VW\}^3,$$

which may be written in the symmetrical form

$$27\,(1 - xyvw)^2\,\{x^3y^3v^3w^3 - (y^3v^3w^3 + v^3w^3x^3 + x^3y^3v^3 + x^3y^3w^3) + xyvw\,(x^3 + y^3 + v^3 + w^3) - xyvw\}\,XYVW$$

$$= \{x^3y^3v^3w^3 - (y^3v^3w^3 + v^3w^3x^3 + x^3y^3v^3 + x^3y^3w^3) + 3x^2y^2v^2w^2 - 3xyvw + x^3 + y^3 + v^3 + w^3 - 1\}^3,$$

which therefore is the general integral in a rational form of the differential equation

$$(1 - x^3)^{-\frac{2}{3}}dx + (1 - y^3)^{-\frac{2}{3}}dy + (1 - v^3)^{-\frac{2}{3}}dv = 0;$$

w being the arbitrary constant.

When $w=\infty$,

$$27x^3y^3v^3(1-x^3)(1-y^3)(1-v^3)=\{1-y^3v^3-v^3x^3-x^3y^3+x^3y^3v^3\}^3 \ldots (2),$$

when $w=1$,

$$(1-x^3)(1-y^3)(1-v^3)=(1-xyv)^3, \ldots\ldots\ldots (3),$$

when $w=0$,

$$27x^3y^3v^3(1-x^3)(1-y^3)(1-v^3)=\{1-x^3-y^3-v^3+x^3y^3v^3\}^3 \ldots\ldots (4),$$

viz. these are particular integrals of the differential equation.

Also each of these last three equations may be looked upon as a general integral of the differential equation

$$(1-x^3)^{-\frac{2}{3}}dx+(1-y^3)^{-\frac{2}{3}}dy=0,$$

and $v=1$ in (2), $v=\infty$ in (3), $v=0$ in (4),

lead to $x^3+y^3=1$,

$v=0$ in (2), $v=1$ in (3), $v=\infty$ in (4),

lead to $xy=1$,

$v=\infty$ in (2), $v=0$ in (3), $v=1$ in (4),

lead to $x^3+y^3=x^3y^3$.

§§ 3. Consider the differential equation

$$\frac{dx}{(1-x^6)^{\frac{5}{6}}}+\frac{dy}{(1-y^6)^{\frac{5}{6}}}=0.$$

Its rational algebraical integral may be found as a particular integral of the equation

$$\frac{dx}{(1-x^6)^{\frac{5}{6}}}+\frac{dy}{(1-y^6)^{\frac{5}{6}}}+\frac{dz}{(1-z^6)^{\frac{5}{6}}}=0.$$

put

$$v_1^3=\frac{1}{4x^6(1-x^6)}$$

$$v_2^3=\frac{1}{4y^6(1-y^6)}$$

$$v_3^3=\frac{1}{4z^6(1-z^6)},$$

then

$$\frac{dv_1}{\sqrt{(v_1^3-1)}}+\frac{dv_2}{\sqrt{(v_2^3-1)}}+\frac{dv_3}{\sqrt{(v_3^3-1)}}=0,$$

a particular integral of which is

$$3v_1v_2v_3 - 3 = \sqrt{(v_2^3 - 1)}\,\sqrt{(v_3^3 - 1)} + \sqrt{(v_3^3 - 1)}\,\sqrt{(v_1^3 - 1)} + \sqrt{(v_1^3 - 1)}\,\sqrt{(v_2^3 - 1)},$$

or $$3\,(v_1v_2v_3 - 1) = (1 - 2y^6)\,(1 - 2z^6)\,\sqrt{(v_2^3v_3^3)} + (1 - 2z^6)\,(1 - 2x^6)\,\sqrt{(v_3^3v_1^3)} + (1 - 2x^6)\,(1 - 2y^6)\,\sqrt{(v_1^3v_2^3)} = a\,\sqrt{(v_2^3v_3^3)} + b\,\sqrt{(v_3^3v_1^3)} + c\,\sqrt{(v_1^3v_2^3)};$$

where, for brevity, we write

$$a=(1-2y^6)(1-2z^6), \quad b=(1-2z^6)(1-2x^6), \quad c=(1-2x^6)(1-2y^6).$$

To rationalise this equation, put

$$t_1 = \frac{1}{v_1}, \qquad t_2 = \frac{1}{v_2}, \qquad t_3 = \frac{1}{v_3},$$

so that $t_1^3 = 4x^6(1 - x^6)$, $t_2^3 = 4y^6(1 - y^6)$, $t_3^3 = 4z^6(1 - z^6)$.

Then

$$3\,\sqrt{(t_1t_2t_3)}\,(1 - t_1t_2t_3) = a\,\sqrt{(t_1^3)} + b\,\sqrt{(t_2^3)} + c\,\sqrt{(t_3^3)} \ldots\ldots (5);$$

and, squaring each side,

$$9t_1t_2t_3\,(1 - t_1t_2t_3)^2 = a_1 + 2bc\,\sqrt{(t_2^3t_3^3)} + 2ca\,\sqrt{(t_3^3t_1^3)} + 2ab\,\sqrt{(t_1^3t_2^3)},$$

where a_1 is rational,

$$= a^2t_1^3 + b^2t_2^3 + c^2t_3^3;$$

transposing a_1, and again squaring,

$$\{9t_1t_2t_3\,(1 - t_1t_2t_3)^2 - a_1\}^2 = b_1 + 8abc\,\sqrt{(t_1^3t_2^3t_3^3)}\,\{a\,\sqrt{(t_1^3)} + b\,\sqrt{(t_2^3)} + c\,\sqrt{(t_3^3)}\};$$

where b_1 is rational,

$$= 4\,\{b^2c^2t_2^3t_3^3 + c^2a^2t_3^3t_1^3 + a^2b^2t_1^3t_2^3;$$

whence, from equation (5),

$$\{9t_1t_2t_3\,(1 - t_1t_2t_3)^2 - a_1\}^2 = b_1 + 24abct_1^2t_2^2t_3^2\,(1 - t_1t_2t_3);$$

writing now $$t_1^3t_3^3t_3^3 = m,$$

expanding the left-hand side and arranging,

$$81 - 324m + 36a_1 - 24abc)\,m^{\frac{2}{3}} + (486m - 18a_1)\,m^{\frac{1}{3}} + 81m^2 - 324m - 18a_1m + a_1^2 - b_1 + 24abcm = 0,$$

or say, for brevity,

$$Am^{\frac{2}{3}} + Bm^{\frac{1}{3}} + C = 0,$$

where $A = 81 - 324m + 36a_1 - 24abc,$

$B = 486m - 18a_1,$

$C = 81m^2 - 324m - 18a_1m + a_1^2 - b_1 + 24abcm$;

transposing C and cubing

$$A^3m^2 + B^2m + 3ABm(Am^{\frac{2}{3}} + Bm^{\frac{1}{3}}) = -C^3,$$

or

$$A^3m^2 + B^3m + C^3 = 3ABCm \quad \ldots\ldots\ldots\ldots \quad (2),$$

which is rational in x, y, and z.

Now

$$\begin{aligned} A &= 81 - 324t_1^3t_2^3t_3^3 + 36\{t_1^3(1-t_2^3)(1-t_3^3) + t_2^3(1-t_3^3)(1-t_1^3) \\ &\qquad + t_3^3(1-t_1^3)(1-t_2^3)\} - 24(1-t_1^3)(1-t_2^3)(1-t_3^3) \\ &= 57 + 60(t_1^3 + t_2^3 + t_3^3) - 96(t_2^3t_3^3 + t_3^3t_1^3 + t_1^3t_2^3) - 192t_1^3t_2^3t_3^3 \\ &= 57 + 60\alpha - 96\beta - 192\gamma \\ &= 3(19 + 20\alpha - 32\beta - 64\gamma), \end{aligned}$$

where

$$\alpha = t_1^3 + t_2^3 + t_3^3 = 4\{x^6(1-x^6) + y^6(1-y^6) + z^6(1-z^6)\},$$

$$\beta = 16\{y^6z^6(1-y^6)(1-z^6) + z^6x^6(1-z^6)(1-x^6) + x^6y^6(1-x^6)(1-y^6)\},$$

$$\gamma = 64x^6y^6z^6(1-x^6)(1-y^6)(1-z^6).$$

Moreover,

$$\begin{aligned} B &= 486t_1^3t_2^3t_3^3 - 18\{t_1^3 + t_2^3 + t_3^3 - 2(t_2^3t_3^3 + t_3^3t_1^3 + t_1^3t_2^3) + 3t_1^3t_2^3t_3^3\} \\ &= 18(-\alpha + 2\beta + 24\gamma); \end{aligned}$$

and since

$$\begin{aligned} b_1 &= 4(b^2c^2t_2^3t_3^3 + c^2a^2t_3^3t_1^3 + a^2b^2t_1^3t_2^3) \\ &= 4(1-t_1^3)(1-t_2^3)(1-t_3^3)(t_2^3t_3^3 + t_3^3t_1^3 + t_1^3t_2^3 - 3t_1^3t_2^3t_3^3), \end{aligned}$$

we have

$$\begin{aligned} C &= 81t_1^6t_2^6t_3^6 - 324t_1^3t_2^3t_3^3 \\ &\quad - 18t_1^3t_2^3t_3^3\{t_1^3 + t_2^3 + t_3^3 - 2(t_2^3t_3^3 + t_3^3t_1^3 + t_1^3t_2^3) + 3t_1^3t_2^3t_3^3\} \\ &\qquad + \{t_1^3 + t_2^3 + t_3^3 - 2(t_2^3t_3^3 + t_3^3t_1^3 + t_1^3t_2^3) + 3t_1^3t_2^3t_3^3\}^2 \\ &\quad - 4(1-t_1^3)(1-t_2^3)(1-t_3^3)\{t_2^3t_3^3 + t_3^3t_1^3 + t_1^3t_2^3 - 3t_1^3t_2^3t_3^3\} \\ &\quad + 24t_1^3t_2^3t_3^3(1-t_1^3)(1-t_2^3)(1-t_3^3) \\ &= 81\gamma^2 - 324\gamma + (\alpha - 2\beta + 3\gamma)(\alpha - 2\beta - 15\gamma) \\ &\qquad + 4(1 - \alpha + \beta - \gamma)(9\gamma - \beta), \end{aligned}$$

or, on reduction,

$$= \alpha^2 - 4\beta - 288\gamma - 48\gamma\alpha + 64\beta\gamma.$$

Therefore, substituting in (2)

$$27\gamma^2 (19 + 20\alpha - 32\beta - 64\gamma)^3 + 5832\gamma (-\alpha + 2\beta + 24\gamma)^3 + (\alpha^2 - 4\beta - 288\gamma - 48\gamma\alpha + 64\beta\gamma)^3$$
$$= 162\gamma (19 + 20\alpha - 32\beta - 64\gamma) (-\alpha + 2\beta + 24\gamma) \times (\alpha^2 - 4\beta - 288\gamma - 48\gamma\alpha + 64\beta\gamma);$$

which, therefore, is the general rational algebraical integral of the differential equation

$$(1 - x^6)^{-\frac{5}{6}} dx + (1 - y^6)^{-\frac{5}{6}} dy = 0,$$

where

$$\alpha = 4 \{x^6 (1 - x^6) + y^6 (1 - y^6) + z^6 (1 - z^6)\},$$

$$\beta = 16 \{y^6 z^6 (1 - y^6)(1 - z^6) + z^6 x^6 (1 - z^6)(1 - x^6) + x^6 y^6 (1 - x^6)(1 - y^6)\},$$

$$\gamma = 64 x^6 y^6 z^6 (1 - x^6) (1 - y^6) (1 - z^6);$$

z being the arbitrary constant, and such that

$$\frac{dx}{(1 - x^6)^{\frac{5}{6}}} + \frac{dy}{(1 - y^6)^{\frac{5}{6}}} + \frac{dz}{(1 - z^6)^{\frac{5}{6}}} = 0.$$

Two particular integrals may be deduced; if

$$z = 0, \text{ or } z = 1,$$

then $$\gamma = 0,$$

and $$\alpha^2 = 4\beta;$$

therefore $$\{x^3 (1 - x^6) + y^6 (1 - y^6)\}^2 = 4 x^6 y^6 (1 - x^6) (1 - y^6),$$

or, on reduction, $$x^6 + y^6 = 1,$$

a particular integral.

Again if $z = \infty$, equating the coefficient of the highest power of z to zero, we find

$$64^2 (1 - x^6) (1 - y^6) \{x^6 (1 - x^6) + y^6 (1 - y^6)\} - 12 . 64 x^6 y^6 (1 - x^6) (1 - y^6) + 1 = 0,$$

a particular integral.

Arranged in ascending powers of x and y this result takes the form

$$1 - 64^2 (x^6 + y^6) - 4 . 64 (32 x^{12} + 35 x^6 y^6 + 32 y^{12}) + 4 . 64 (x^6 + y^6) (16 x^{12} + 19 x^6 y^6 + 16 y^{12}) - 4 . 64 x^6 y^6 (16 x^{12} + 3 x^6 y^6 + 16 y^{12}) = 0.$$

The locus represented by the general equation is of the 144th degree in x and y: moreover the equation is a symmetrical equation of degree 72 in each of the coordinates x and y separately.

Royal Military Academy,
December 20th, 1882.

NOTE ON THE FOREGOING PAPER.

By Prof. CAYLEY.

IN general if $f, = (x, y, 1)^3, = 0$ be the equation of a cubic curve, and if $d\omega = \dfrac{dx}{\dfrac{df}{dy}}, = \dfrac{-dy}{\dfrac{df}{dx}}$, then if 1, 2, 3 are the intersections of the curve by an arbitrary right line, the coordinates of these points being (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively, we have, by Abel's theorem,

$$d\omega_1 + d\omega_2 + d\omega_3 = 0,$$

viz. this is the differential relation corresponding to the integral relation which expresses that the three points are the intersections of the cubic curve by a right line, or say to the integral equation

$$\nabla, = \begin{vmatrix} y_1, & y_2, & y_3 \\ x_1, & x_2, & x_3 \\ 1, & 1, & 1 \end{vmatrix}, = 0,$$

in which equation y_1, y_2, y_3 are regarded as functions of x_1, x_2, x_3 respectively, given by means of the equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ which express that the points are on the cubic curve. See my "Memoir on the Abelian and Theta Functions," about to be published in the *American Journal of Mathematics*, vol. v.

In particular if the equation of the curve is

$$f, = \tfrac{1}{3}\{y^3 - (A + 3Bx + 3Cx^2 + Dx^3)\}, = \tfrac{1}{3}(y^3 - X), = 0,$$

then

$$d\omega = \frac{dx}{y^2}, = \frac{dx}{X^{\frac{2}{3}}};$$

and corresponding to the differential relation

$$X_1^{-\frac{2}{3}}\, dx_1 + X_2^{-\frac{2}{3}}\, dx_2 + X_3^{-\frac{2}{3}}\, dx_3 = 0,$$

we have the integral relation

$$\nabla, \quad = \begin{vmatrix} X_1^{\frac{1}{3}}, & X_2^{\frac{1}{3}}, & X_3^{\frac{1}{3}} \\ x_1, & x_2, & x_3 \\ 1, & 1, & 1 \end{vmatrix} = 0,$$

viz., this last equation as containing no arbitrary constant, is a particular integral of the differential equation.

If instead of x_1, x_2, x_3 we write x, y, z, then we have

$$\nabla, \quad = \begin{vmatrix} X^{\frac{1}{3}}, & Y^{\frac{1}{3}}, & Z^{\frac{1}{3}} \\ x, & y, & z \\ 1, & 1, & 1 \end{vmatrix}, \quad = 0$$

as a particular integral of the differential equation

$$X^{-\frac{2}{3}}\,dx + Y^{-\frac{2}{3}}\,dy + Z^{-\frac{2}{3}}\,dz = 0.$$

To rationalize the integral equation, write

$$\alpha, \beta, \gamma = y - z, z - x, x - y \text{ (so that } \alpha + \beta + \gamma = 0),$$

the equation is

$$\alpha X^{\frac{1}{3}} + \beta Y^{\frac{1}{3}} + \gamma Z^{\frac{1}{3}} = 0;$$

and we thence have

$$\alpha^3 X + \beta^3 Y + \gamma^3 Z = 3\alpha\beta\gamma X^{\frac{1}{3}} Y^{\frac{1}{3}} Z^{\frac{1}{3}}.$$

The left hand side is

$$\begin{aligned} &\alpha^3 (A + 3Bx + 3Cx^2 + Dx^3) \\ &+ \beta^3 (A + 3By + 3Cy^2 + Dy^3) \\ &+ \gamma^3 (A + 3Bz + 3Cz^2 + Dz^3); \end{aligned}$$

or assuming

$$\alpha', \beta', \gamma' = x(y - z), y(z - x), z(x - y) \text{ (so that } \alpha' + \beta' + \gamma' = 0),$$

this is $= A(\alpha^3 + \beta^3 + \gamma^3) + 3B(\alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma')$

$$+ 3C(\alpha\alpha'^2 + \beta\beta'^2 + \gamma\gamma'^2) + D(\alpha'^3 = \beta'^3 + \gamma'^3).$$

But taking λ arbitrary, and

$$a, b, c = \alpha + \lambda\alpha', \ \beta + \lambda\beta', \ \gamma + \lambda\gamma',$$

then $$a + b + c = 0,$$

whence $$a^3 + b^3 + c^3 = 3abc;$$

or substituting for a, b, c their values, and comparing the coefficients of the several powers of λ,

$$\begin{aligned} \alpha^3 + \beta^3 + \gamma^3 &= 3\alpha\beta\gamma, \\ \alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma' = \alpha'\beta\gamma + \beta'\gamma\alpha + \gamma'\alpha\beta &= \alpha\beta\gamma(x + y + z), \\ \alpha\alpha'^2 + \beta\beta'^2 + \gamma\gamma'^2 = \alpha\beta'\gamma' + \beta\gamma'\alpha' + \gamma\alpha'\beta' &= \alpha\beta\gamma(yz + zx + xy), \\ \alpha'^3 + \beta'^3 + \gamma'^3 = 3\alpha'\beta'\gamma' &= 3\alpha\beta\gamma . xyz. \end{aligned}$$

Hence we have

$$\alpha^3X+\beta^3Y+\gamma^3Z=3\alpha\beta\gamma\{A+B(x+y+z)+C(yz+zx+xy)+Dxyz\},$$

or the integral equation is

$$\{A+B(x+y+z)+C(yz+zx+xy)+Dxyz\}=X^{\frac{1}{3}}Y^{\frac{1}{3}}Z^{\frac{1}{3}},$$

that is

$$\{A+B(x+y+z)+C(yz+zx+xy)+Dxyz\}^3=XYZ,$$

the elegant result given by Capt. MacMahon at the beginning of his paper.

The author in a letter to me, dated Jan. 13, 1883, remarks that the particular integral of the equation in question

$$X^{-\frac{2}{3}}dx+Y^{-\frac{2}{3}}dy+Z^{-\frac{2}{3}}dz=0,$$

is expressible as a determinant in a rational form as follows: Writing it $XYZ=P^3$, where

$$P=A+B(x+y+z)+C(yz+zx+xy)+Dxyz,$$

then the form is

$$\nabla, = \begin{vmatrix} 1, & \left(\frac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)^3, & X \\ 1, & \left(\frac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)^3, & Y \\ 1, & \left(\frac{1}{3}P\frac{dZ}{dz}-Z\frac{dP}{dz}\right)^3, & Z \end{vmatrix}=0,$$

for, as shown *ante*, p. 171, each of the three terms such as

$$Z\left(\frac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)^3-Y\left(\frac{1}{3}P\frac{dZ}{dz}-Z\frac{dP}{dz}\right)^3$$

which compose the determinant is divisible by $XYZ-P^3$.

It may be added, that we have identically

$$\frac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}=(AC-B^2)(2x-y-z)$$
$$+(AD-BC)(x^2-yz)+(BD-C^2)(x^2y+x^2z-2xyz),$$

and of course like values for the other two expressions in the determinant.

NOTE ON THE SPHERICAL TRIANGLE IN ELLIPTIC FUNCTIONS.

By Professor William Woolsey Johnson.

§ 1. IN the proof of the addition theorem in Elliptic Functions by means of a spherical triangle whose sides are ϕ, ψ, and μ, where

$$\phi = \operatorname{am} u, \qquad \psi = \operatorname{am} v, \qquad \mu = \operatorname{am}(u+v),$$

and k is the ratio of the sines of the angles to the sines of the opposite sides, it is usual to state that the angle opposite to the side μ is obtuse, so that its cosine is $-\Delta\mu$, if the other angles are acute so that their cosines are $\Delta\phi$ and $\Delta\psi$. This may indeed be shown to be a consequence of the assumption that k is less than unity; the theorem being, that in a spherical triangle in which the sines of the angles are less than the sines of the opposite sides, either one of the angles must be obtuse and the other two acute, or all three must be obtuse.* It is the object of this note to show that we may remove the restriction $k<1$, in accordance with which $\Delta\mu$ is always positive; proving directly that in all cases the cosine of the angle in question is $-\Delta\mu$. Moreover, it seems desirable to make the proof as complete as possible, by showing that the triangle from which the formulæ are derived, is possible for all real values of u and v as well as of k.

§ 2. If $\phi = \operatorname{am} u$, where

$$u = \int_0 \frac{dx}{\sqrt{(1-x^2)}\,\sqrt{(1-k^2x^2)}},$$

we have $\qquad \sin\phi = x \quad$ and $\quad \cos\phi = \sqrt{(1-x^2)}$;

let ϕ' be an angle such that

$$\sin\phi' = kx \quad \text{and} \quad \cos\phi' = \sqrt{(1-k^2x^2)};$$

ϕ' like ϕ being assumed to vanish with u. Confining our attention to real values of u, if $k<1$, x is restricted to values between $+1$ and -1; and in order that u may increase continuously, x must increase and decrease alternately, the radical $\sqrt{(1-x^2)}$ changing sign as well as dx, whenever x passes through its maximum or minimum value. Thus ϕ increases continuously, passing through the values $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, &c. when x passes periodically through its extreme values; but ϕ' is restricted to values between $\pm \operatorname{arc}\sin k$, its cosine re-

* *Quarterly Journal*, vol. xvii. p. 355.

maining always positive. On the other hand, when $k>1$, x is restricted to values between $+\frac{1}{k}$ and $-\frac{1}{k}$; and as u increases continuously, ϕ' increases continuously, while ϕ is restricted to values between $\pm$ arc sin $\frac{1}{k}$, its cosine remaining always positive.

Thus, in the first case, ϕ admits of all values and ϕ' is restricted; and in the second case ϕ' admits of all values and ϕ is restricted.

§ 3. In accordance with this notation, we have then

$$\begin{aligned} \sin\phi &= \operatorname{sn} u, & \sin\psi &= \operatorname{sn} v,\\ \cos\phi &= \operatorname{cn} u, & \cos\psi &= \operatorname{cn} v,\\ \sin\phi' &= k\operatorname{sn} u, & \sin\psi' &= k\operatorname{sn} v,\\ \cos\phi' &= \Delta\phi = \operatorname{dn} u, & \cos\psi' &= \Delta\psi = \operatorname{dn} v. \end{aligned}$$

Let u_0 and v_0 be any initial values of u and v, then ϕ_0, ϕ'_0, ψ_0, ψ'_0 are real angles, and it is possible to construct a spherical triangle, of which two sides are ϕ_0 and ψ_0, and the opposite angles ϕ_0' and ψ_0'; for if we construct the triangle in which $BC=\phi_0$, $CA=\psi_0$, and $CBA=\psi_0'$, the sine of the angle opposite to BC will be $k\sin\phi_0$, which is $\sin\phi_0'$, and of the two possible triangles we may select that in which the angle at A is ϕ_0' and not $\pi-\phi_0'$. Denoting the side AB (fig. 15) of the triangle thus constructed by μ, and permitting the triangle to vary from its initial form in such a manner that μ and the opposite angle C remain constant, u and v become variables, and the differential relation

$$\frac{d\phi}{\cos\phi'}+\frac{d\psi}{\cos\psi'}=0,$$

shows that

$$u+v=w,$$

where w is a constant. When in the variation of the triangle ψ and ψ' vanish, ϕ becomes equal to μ and ϕ' to the *exterior* angle at C, and since when $v=0$, $u=w$, we have $\mu=\operatorname{am} w$, and the exterior angle is μ', so that

$$\begin{aligned} \sin\mu &= \operatorname{sn} w = \operatorname{sn}(u+v),\\ \cos\mu &= \operatorname{cn}(u+v),\\ \sin\mu' &= k\operatorname{sn}(u+v),\\ \cos\mu' &= \Delta\mu = \operatorname{dn}(u+v), \end{aligned}$$

and the relations between the parts of the triangle give the various forms of the addition theorem.

§ 4. When $k<1$, the angles ϕ', ψ' and μ' are all acute; that is, one angle of the constructed triangle is obtuse. On the other hand, when $k>1$, the three sides ϕ, ψ, and μ are all in the first quadrant, the angles being unrestricted. The theorem in this case is, that in an ordinary spherical triangle in which the sines of the angles are greater than the sines of the opposide sides, either all three of the sides are in the first quadrant, or else one is in the first and two are in the second quadrant.

§ 5. It may be remarked, that since in the other triangle in which $BC=\phi$, $CA=\psi$, and $CBA=\psi'$, ϕ' is the exterior angle at A, this triangle corresponds to the relation

$$u=v+w;$$

from which it follows at once that, if a perpendicular be dropped from C, the segments of the base are

$$\tfrac{1}{2}\operatorname{am}(u+v)\pm\tfrac{1}{2}\operatorname{am}(u-v).$$

§ 6. There are of course a variety of other triangles corresponding to the relation

$$u+v=w,$$

which may be derived from that constructed in § 3 or from one another, either by transferring one vertex to the diametrically opposite point of the sphere, that is, by replacing two of the sides and their opposite angles by their supplements, or by substituting for one side the remainder of the great circle of which it forms a part, that is, by replacing a side and its opposite angle by their negatives, and the other two angles by their supplements. Thus instead of

sides	ϕ,	ψ,	μ,
angles	ϕ',	ψ',	$\pi-\mu'$,

we may have for example

sides	$\pi-\phi$,	$\pi-\psi$,	μ,
angles	$\pi-\phi'$,	$\pi-\psi'$,	$\pi-\mu'$,
or sides	$\pi-\phi$,	$\pi+\psi$,	μ,
angles	ϕ',	$\pi+\psi'$,	μ'.

§ 7. We might also employ a triangle in which

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=k;$$

in fact, the polar triangle of ABC in the diagram has the

sides	$\pi-\phi'$,	$\pi-\psi'$,	μ',
angles	$\pi-\phi$,	$\pi-\psi$,	$\pi-\mu$,

and this we may replace by the triangle having the

sides	ϕ',	ψ',	μ',
angles	ϕ,	ψ,	$\pi-\mu$.

§ 8. If ϕ, ψ, and θ are the amplitudes of three arguments satisfying the symmetrical relation

$$u+v+w=0,$$

we have in the triangle ABC of the diagram, since $\mu=-\theta$ and $\mu'=-\theta'$, the

sides	ϕ,	ψ,	$-\theta$,
angles	ϕ',	ψ',	$\pi+\theta$,

and, as in § 6, we may replace this by the triangle having the

sides	ϕ,	ψ,	θ,
angles	$\pi-\phi'$,	$\pi-\psi'$,	$\pi-\theta'$.

We may also employ the polar triangle having

sides	ϕ',	ψ',	θ',
angles	$\pi-\phi$,	$\pi-\psi$,	$\pi-\theta$.

Denoting sn, cn, dn, by s, c, d, and using suffixes to indicate the arguments, we see that we may in any formula of spherical trigonometry put

$\sin a =$	s_1	or	$\sin a =$ ks_1,
$\cos a =$	c_1	,,	$\cos a =$ d_1,
$\sin A =$	ks_1	,,	$\sin A =$ s_1,
$\cos A =$	$-d_1$	,,	$\cos A =$ $-c_1$,
&c.,			&c.,

where $u+v+w=0$. In either case the system of sixteen formulæ corresponding to those given on pp. 366 and 368 of vol. XVIII. for the case $u+v=w$, become

three of the form	$c_1=c_2c_3-s_2s_3d_1$,
three of the form	$d_1=d_2d_3-k^2s_2s_3c_1$,
six of the form	$s_1c_2d_3+s_2c_1+s_3d_1=0$,

and the single formula

$$k^2=\frac{1-d_1d_2d_3}{1-c_1c_2c_3}.$$

NOTICE.

HERR CARL SIEWERS communicates the following notice of a new Scandinavian Mathematical Journal:

The Acta Mathematica.

The long-felt want in Scandinavia of a mathematical journal will soon be supplied, as it is intended by mathematicians to start a journal called *The Acta Mathematica*, the object of which is to gather the mathematical forces and form a connecting link between mathematicians in the Scandinavian countries, and lay the results of their labours before the scientific world. The journal will be specially devoted to the publication of valuable results and new methods in the treating of problems.

As Editor-in-chief has been selected Herr G. Mittag-Leffler, Professor at the 'High-school' in Stockholm, while among the assistant editors are Professor H. Gyldén and H. Holmgren, of Stockholm; H. T. Dauz and C. J. Malmsten, of Upsala; A. W. Bäcklund, of Lund; O. J. Broch, C. A. Bjerknes and S. Lie, of Christiania; Rector L. Sylow, of Fredrikstad; Professors L. Lorenz, J. Petersen and H. G. Zeuthen, of Copenhagen; and Herr L. Lindelöf, of Helsingfors. The Publishers are Messrs. F. and G. Beijer, Stockholm; Meyer and Müller, Berlin; and A. Herman, Paris. The financial position of the work has been ensured by several persons, the King of Sweden, among others, having offered a yearly sum for three years, while State grants will probably be given from Sweden, Norway, Denmark, and Finland.

Contributions have been offered by eminent mathematicians, as, for instance, Hermite, Poincaré, Piccard and Appell, of Paris; Weierstrass and Fuchs, of Berlin; Schering, of Gottingen; and Casorati, of Pavia, &c. The papers will be written in French or German, and, exceptionally, also in English and Latin.

The Acta Mathematica will appear periodically, and cost for England 12 *sh.* per volume of about 50 printed sheets.

No. I of the *Journal* has appeared, and contains Poincaré, Théorie des groupes Fuchsiens, pp. 1–63; Malmsten, zur Théorie der Leibrenten, pp. 63–76; Gyldén, eine Annaherung's methode in Probleme der drei Körper, pp. 77–92; Reye, das Problem der Configurationem, pp. 93–96.

ON THE MOTION OF A LIQUID IN AND ABOUT CYLINDERS WHOSE TRANSVERSE SECTIONS ARE THE INVERSE OF CONFOCAL ELLIPSES WITH RESPECT TO THEIR CENTRE.

By A. B. Basset, M.A.

IF $$\xi + \iota\eta = \sec^{-1}\frac{x+\iota y}{c},$$

then $$r^4 = c^2\left(\frac{x^2}{\cosh^2\eta} + \frac{y^2}{\sinh^2\eta}\right),$$

and $$r^4 = c^2\left(\frac{x^2}{\cos^2\xi} - \frac{y^2}{\sin^2\xi}\right),$$

so that the curves $\eta = \beta$ and $\xi = \alpha$ are the inverse of a family of confocal ellipses and hyperbolas.

We shall call the two points F, F' (see fig. 16), whose distances from the origin are respectively + and $-c$, and which are the inverse points of the foci of the confocal ellipses and hyperbolas, the foci of the system.

Before proceeding to discuss any of the physical problems connected with these curves, it will be necessary to examine the forms which they assume for different values of the parameters η and ξ; the values of η being restricted to all positive *real* values between zero and infinity, whilst ξ may have any *real* value positive or negative.

Writing the curve $\eta = \beta$ in the form

$$r^4 = a^2x^2 + b^2y^2,$$

we find $$\tan\psi = \frac{dy}{dx} = -\frac{x}{y}\,\frac{2r^2 - a^2}{2r^2 - b^2}$$

$$= -\cot\theta\,\frac{b^2 + (a^2 - b^2)\cos 2\theta}{a^2 + (a^2 - b^2)\cos 2\theta}.$$

Taking logarithms and differentiating, we find that ψ will be a minimum when

$$\cos 2\theta = \frac{a^4 - a^2b^2 + b^4}{b^4 - a^4}.$$

In order that the value of θ may be real, we must have

$$b^4 - a^4 > a^4 - a^2b^2 + b^4,$$

or
$$b^2 > 2a^2,$$

or
$$\eta < \tfrac{1}{2} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Hence, if this condition is satisfied, there will be four points of inflexion at the points θ, $\pi - \theta$, $\pi + \theta$ and $2\pi - \theta$, where θ has the value obtained above.

When η is very large the curves $\eta = \text{const.}$ are small, and nearly circular curves about the origin, with which they alternately coincide when $\eta = \infty$. As η diminishes, the curves assume an oval form, the longest diameter or major-axis of which coincides with the axis of y, whilst the shortest diameter or minor-axis coincides with the axis of x. As soon as η becomes less than $\frac{1}{2} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$ there are four points of inflexion as we have seen, and the curve begins to double back upon the axis of x, whilst the major-axis continues to elongate; the limiting form, when $\eta = 0$, consists of two doubled lines extending from the foci to infinity in the positive and negative directions of the axis of x respectively.

The curves $\xi = \text{const.}$ consist of four distinct portions which correspond to the values $2n\pi + \alpha$, $(4n+1)\frac{1}{2}\pi + \alpha$, $(2n+1)\pi + \alpha$, and $(4n+3)\frac{1}{2}\pi + \alpha$ of ξ, and lie in the four respective quadrants. These four portions together form a figure of eight, whose centre is the origin and which cuts the axis of x at the two points $\pm c \sec \xi$. As ξ decreases the breadth of the loops diminish, and when $\xi = 0$ the curve ultimately becomes two doubled lines drawn from origin to each of the two foci. On the other hand, when $\xi = \frac{1}{2}\pi$, the curve consists of two straight lines coinciding with the axis of y.

A drawing of the curves is given in fig. 16.

We shall in the next place prove certain formulæ relating to these curves: we have

$$\xi + \iota\eta = \sec^{-1} \frac{x + cy}{c},$$

therefore
$$\frac{cx}{r^2} = \cosh\eta \cos\xi,$$

$$\frac{cy}{r^2} = \sinh\eta \sin\xi,$$

$$\frac{2c^2}{r^2} = \cosh 2\eta + \cos 2\xi,$$

also, since $$\frac{d\xi}{dx} = \frac{d\eta}{dy} \text{ and } \frac{d\xi}{dy} = -\frac{d\eta}{dx},$$

$$\cosh\eta \sin\xi \frac{d\xi}{dx} + \sinh\eta \cos\xi \frac{d\xi}{dy} = \frac{1}{2c}(1 + \cosh 2\eta \cos 2\xi),$$

$$\sinh\eta \cos\xi \frac{d\xi}{dx} - \cosh\eta \sin\xi \frac{d\xi}{dy} = -\frac{1}{2c}\sinh 2\eta \sin 2\xi;$$

therefore

$$\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 = J^2 = \frac{(\cosh 2\eta + \cos 2\xi)^2}{2c^2(\cosh 2\eta - \cos 2\xi)},$$

and $$\frac{d\xi}{dx} = \frac{d\eta}{dy} = \frac{1}{c}\left(1 + \frac{2}{\cosh 2\eta - \cos 2\xi}\right)\cosh\eta \sin\xi,$$

$$\frac{d\xi}{dy} = -\frac{d\eta}{dx} = \frac{1}{c}\left(1 + \frac{2}{\cosh 2\eta - \cos 2\xi}\right)\sinh\eta \cos\xi.$$

Writing the curve $\eta = \beta$ in the form

$$r^4 = a^2x^2 + b^2y^2,$$

the equation to the tangent at x, y is

$$xx'(2r^2 - a^2) + yy'(2r^2 - b^2) = r^4,$$

and if p be the perpendicular on the tangent, and l, m its direction cosines,

$$\frac{l}{x(2r^2 - a^2)} = \frac{m}{y(2r^2 - b^2)} = \frac{p}{r^4};$$

therefore $$\frac{p}{r^4} = \frac{1}{\sqrt{(a^4x^2 + b^4y^2)}},$$

therefore $$p = \frac{r^2\sqrt{2}\cosh\beta \sinh\beta}{c\sqrt{(\cosh 2\beta - \cos 2\xi)}} = \frac{Jr^4}{2c^2}\sinh 2\beta.$$

Let R be the distance between the two points x, y and x', y'. Then

$$\begin{aligned}
R^2 &= (x - x')^2 + (y - y')^2 \\
&= \{x + \iota y - (x' + \iota y')\}\{x - \iota y - (x' - \iota y')\} \\
&= \frac{r^2r'^2}{c^2}\{\cos(\xi - \iota\eta) - \cos(\xi' - \iota\eta')\}\{\cos(\xi + \iota\eta) - \cos(\xi' + \iota\eta')\} \\
&= \frac{4r^2r'^2}{c^2}\sin\tfrac{1}{2}\{\xi' + \xi - \iota(\eta' + \eta)\}\sin\tfrac{1}{2}\{\xi' - \xi - \iota(\eta' - \eta)\} \\
&\qquad \times \sin\tfrac{1}{2}\{\xi' + \xi + \iota(\eta' + \eta)\}\sin\tfrac{1}{2}\{\xi' - \xi + \iota(\eta' - \eta)\} \\
&= \frac{4c^2\{\cosh(\eta' + \eta) - \cos(\xi' + \xi)\}\{\cosh(\eta' - \eta) - \cos(\xi' - \xi)\}}{(\cosh 2\eta + \cos 2\xi)(\cosh 2\eta' + \cos 2\xi')};
\end{aligned}$$

also

$$\log\{\cosh(\eta'+\eta)-\cos(\xi'+\xi)\}=\log\tfrac{1}{2}+\eta'+\eta+\log\{1-\varepsilon^{-\eta'-\eta+\iota(\xi'+\xi)}\}+\log\{1-\varepsilon^{-\eta'-\eta-\iota(\xi'+\xi)}\}$$

$$=\log\tfrac{1}{2}+\eta'+\eta-2\sum_1^\infty\frac{1}{n}\varepsilon^{-n(\eta'+\eta)}\cos n(\xi'+\xi),$$

and $\log\{\cosh(\eta'-\eta)-\cos(\xi'-\xi)\}$

$$=\log\tfrac{1}{2}+\eta'-\eta-2\sum_1^\infty\frac{1}{n}\varepsilon^{-n(\eta'-\eta)}\cos n(\xi'-\xi),$$

when the point ξ', η' is within the cylinder η, and

$$=\log\tfrac{1}{2}+\eta-\eta'-2\sum_1^\infty\frac{1}{n}\varepsilon^{-n(\eta-\eta')}\cos x(\xi-\xi'),$$

when this point is without the cylinder, and

$\log(\cosh 2\eta+\cos 2\xi)$

$$=\log\tfrac{1}{2}+2\eta+\log\{1+\varepsilon^{-2(\eta-\iota\xi)}\}+\log\{1+\varepsilon^{-2(\eta+\iota\xi)}\}$$

$$=\log\tfrac{1}{2}+2\eta-2\sum_1^\infty\frac{(-1)^n}{n}\varepsilon^{-2n\eta}\cos 2n\xi;$$

therefore when η', ξ' is within the cylinders

$$\log R=\log 2c+\tfrac{1}{2}\log\tfrac{1}{2}+\tfrac{1}{2}(\eta'+\eta)-\sum_1^\infty\frac{1}{n}\varepsilon^{-n(\eta'-\eta)}\cos n(\xi'+\xi)$$

$$+\tfrac{1}{2}\log\tfrac{1}{2}+\tfrac{1}{2}(\eta'-\eta)-\sum_1^\infty\frac{1}{n}\varepsilon^{-n(\eta'-\eta)}\cos n(\xi'-\xi)$$

$$-\tfrac{1}{2}\log\tfrac{1}{2}-\eta+\sum_1^\infty\frac{(-1)^n}{n}\varepsilon^{-2n\eta}\cos 2n\xi$$

$$-\tfrac{1}{2}\log(\cosh 2\eta'+\cos 2\xi');$$

therefore

$$\log R=\log c\sqrt{2}+\eta'-\eta$$

$$-\sum_1^\infty\frac{1}{n}\left\{\varepsilon^{-n(\eta'+\eta)}\cos n(\xi'+\xi)+\varepsilon^{-n(\eta'-\eta)}\cos n(\xi'-\xi)\right.$$

$$\left.-(-1)^n c^{-2n\eta}\cos 2n\xi\right\}-\tfrac{1}{2}\log(\cosh 2\eta'+\cos 2\xi').$$

For all points within the cylinder, r^2 can be expanded in a series of ascending power of $\varepsilon^{-2\eta}$, for

$$r^2 = \frac{2c^2}{\cosh 2\eta + \cos 2\xi} = \frac{4c^2\varepsilon^{-2\eta}}{\{1+\varepsilon^{-2(\eta-\iota\xi)}\}\{1+\varepsilon^{-2(\eta+\iota\xi)}\}}$$

$$= \frac{2c^2}{\iota \sin 2\xi}\left\{\frac{1}{1+\varepsilon^{-2(\eta+\iota\xi)}} - \frac{1}{1+\varepsilon^{-2(\eta-\iota\xi)}}\right\}$$

$$= \frac{4c^2}{\sin 2\xi}\sum_1^\infty (-1)^{n-1}\varepsilon^{-2n\xi}\sin 2n\xi.$$

Let dn_η, dn_ξ be elements of the normals to the curves η and ξ drawn in the directions of η and ξ *increasing*, then

$$dn_\eta = \frac{d\eta}{J}, \qquad dn_\xi = \frac{d\xi}{J},$$

and if p and q be the velocities in these directions,

$$p = \frac{d\phi}{dn_\eta} = J\frac{d\phi}{d\eta} = \quad J\frac{d\psi}{d\xi},$$

$$q = \frac{d\phi}{dn_\xi} = J\frac{d\phi}{d\xi} = -J\frac{d\psi}{d\eta},$$

since $$\psi + \iota\phi = f(\xi + \iota\eta).$$

As J does not change sign as ξ increases from 0 to 2π, the directions of the velocities depend solely on the signs of the derivatives of ϕ and ψ.

The value of J may be written

$$J = \frac{(1+\varepsilon^{-4\eta}+2\varepsilon^{-2\eta}\cos 2\xi)\,\varepsilon^{\eta}}{2c\,(1+\varepsilon^{-4\eta}-2\varepsilon^{-2\eta}\cos 2\xi)^{\frac{1}{2}}},$$

which becomes infinite at the origin where $\eta = \infty$, but

$$J\varepsilon^{-\eta} = \tfrac{1}{2}c,$$

when $\eta = \infty$, and is therefore finite. Now it will appear hereafter, that the velocity and current functions for motion inside the cylinder can be expressed in the form

$$\psi = \sum_1^\infty \varepsilon^{-n\eta}(A_n\cos n\xi + B_n\sin n\xi),$$

so that the velocities are necessarily finite at the origin.

At a point near the foci $\eta = \delta\eta$, $\xi = \delta\xi$, or $\pi - \delta\xi$, and the value of J is

$$\frac{1}{c} \cdot \frac{1 + \delta\eta^2 - \delta\xi^2}{\sqrt{(\delta\eta^2 + \delta\xi^2)}} = \frac{1}{c} \cdot \frac{1}{\sqrt{(\delta\eta^2 + \delta\xi^3)}},$$

which is infinite at the foci. Hence the condition that the velocities should be finite at the foci is, that in the neighbourhood of these points the derivatives of ϕ and ψ should be small quantities of the first order.

At two points P, P' taken on opposite sides of the line Fx, the velocities p and q are measured in the directions of the arrows (see fig. 17); hence the motion will be discontinuous along Fx, unless the derivatives of ψ either vanish when $\eta = 0$, or change sign in passing from one side to the other.

At points at a great distance from the origin

$$\eta = \delta\eta \text{ and } \xi = \tfrac{1}{2}\pi - \delta\xi,$$

so that

$$J = \frac{1}{c\sqrt{2}} \cdot \frac{\cosh 2\delta\eta - \cos 2\delta\xi}{(\cosh 2\delta\eta + \cos 2\delta\xi)^{\frac{1}{2}}}$$

$$= \frac{1}{c}(\delta\eta^2 + \delta\xi^2).$$

Hence the velocities will vanish at infinity, provided the derivatives of ψ do not become infinite.

It should be observed, that the infinitesimal quantities $\delta\eta$ and $\delta\xi$ are not necessarily of the same order, for if θ be the vectorial angle of a point P very distant from the origin

$$\tan\theta = \frac{y}{x} = \frac{\delta\eta}{\delta\xi},$$

so that when P is near the axis of x, $\delta\xi$ is large in comparison with $\delta\eta$, and the contrary is the case when the point is situated in the neighbourhood of the axis of y.

We shall now proceed to investigate the initial motion of a liquid contained between the two cylinders $\eta = \alpha$ and $\eta = \beta$, when either or both the cylinders are moved in any given manner.

Let the inner cylinder β be moved parallel to the axis of y with velocity V, and let ϕ and ψ be the velocity and current functions of the *initial* motion.

We must determine ψ so that

$$\psi = 0 \quad \text{when} \quad \eta = \alpha,$$

$$\psi = Vx \quad \text{when} \quad \eta = \beta.$$

Now $cx = r^2 \cosh\eta \cos\xi$;

therefore $$x = \frac{2c\cosh\eta}{\sin\xi} \sum_1^\infty (-1)^{n-1} \varepsilon^{-2n\eta} \sin 2n\xi.$$

Assume

$$\psi = \sum_1^\infty A_{2n+1} \frac{\sinh(2n+1)(\eta-\alpha)}{\sinh(2n+1)(\beta-\alpha)} \cos(2n+1)\xi;$$

therefore, when $\eta = \beta$, we must have

$$2\Sigma A_{2n+1} \sin\xi (2n+1)\xi = 4Vc\cosh\beta (\varepsilon^{-2\beta}\sin 2\xi - \varepsilon^{-4\beta}\sin 4\xi + \ldots),$$

now $2\sin\xi\cos(2n+1)\xi = \sin 2(n+1)\xi - \sin 2n\xi$;

therefore $-A_{2n+1} + A_{2n-1} = 4Vc\cosh\beta\,\varepsilon^{-2\beta}(-1)^{n-1}$;

therefore $u_{n+1} - u_n = -4Vc\cosh\beta\,\varepsilon^{-2(n+1)\beta}(-1)^n$,

if $u_n = A_{2n+1}$;

therefore $$u_n = A + \frac{4Vc\cosh\beta\,\varepsilon^{-2(n+1)\beta}(-1)^n}{\varepsilon^{-2\beta+1}},$$

when $n=1$, $u_1 = A_3 = -4Vc\,\varepsilon^{-2\beta}\cosh\beta$;

therefore $$A = -\frac{4Vc\,\varepsilon^{-2\beta}\cosh\beta}{1+c^{-2\beta}};$$

therefore $$A_{2n+1} = -4Vc\,\varepsilon^{-2\beta}\cosh\beta\,\frac{1-(-1)^n\varepsilon^{-2n\beta}}{1+c^{-2\beta}}$$

$$= -2Vc\,\varepsilon^{-\beta}\{1-(-1)^n\varepsilon^{-2n\beta}\};$$

therefore

$$\psi = -2Vc\,\varepsilon^{-\beta}\sum_1^\infty \{1-(-1)^n c^{-2n\beta}\}\frac{\sinh(2n+1)(\eta-\alpha)}{\sinh(2n+1)(\beta-\alpha)}\cos(2n+1)\xi,$$

and

$$\phi = -2Vc\,\varepsilon^{-\beta}\sum_1^\infty \{1-(-1)^n\varepsilon^{-2n\beta}\}\frac{\cosh(2n+1)(\eta-\alpha)}{\sinh(2n+1)(\beta-\alpha)}\sin(2n+1)\xi.$$

If the outer cylinder had begun to move with velocity U parallel to the axis of y, we must assume

$$\psi = Mx + \chi,$$

where M is a constant and χ is a function satisfying the equation $\nabla^2\chi = 0$, and we must determine M and χ, such that

$$\psi = Ux \quad \text{when} \quad \eta = \alpha,$$
$$\psi = 0 \quad \text{when} \quad \eta = \beta,$$

these conditions are satisfied by making $M = U$ and determining χ, such that

$$\chi = 0 \quad \text{when} \quad \eta = \alpha,$$

and

$$\chi = -Ux \quad \text{when} \quad \eta = \beta.$$

Hence

$$\psi = \frac{2Uc\cosh\eta\cos\xi}{\cosh 2\eta + \cos 2\xi}$$

$$+ 2Uc\,\varepsilon^{-\beta}\sum_1^\infty \{1-(-1)^n\varepsilon^{-2n\beta}\}\frac{\sinh(2n+1)(\eta-\alpha)}{\sinh(2n+1)(\beta-\alpha)}\cos(2n+1)\,\xi,$$

and similarly ϕ can be found.

If the inner cylinder had begun to move parallel to the axis of x, we should find

$$\psi = -2Vc\,\varepsilon^{-\beta}\sum_1^\infty (1-\varepsilon^{-2n\beta})(-1)^n\frac{\sinh(2n+1)(\eta-\alpha)}{\sinh(2n+1)(\beta-\alpha)}\sin(2n+1)\,\xi,$$

and

$$\phi = 2Vc\,\varepsilon^{-\beta}\sum_1^\infty (1-\varepsilon^{-2n\beta})(-1)^n\frac{\sinh(2n+1)(\eta-\alpha)}{\sinh(2n+1)(\beta-\alpha)}\cos(2n+1)\,\xi.$$

In considering the motion of the fluid due to the rotation of the cylinders about their axes, it will be convenient for the purpose of avoiding repetition of the figure 17, to put

$$\tfrac{1}{2}(\xi + \iota\eta) = \sec^{-1}\frac{(x+\iota y)}{c},$$

so that

$$r^2 = \frac{4c^2}{\sin\xi}\sum_1^\infty (-1)^{n-1}\varepsilon^{-n\eta}\sin n\xi.$$

If the inner cylinder β begin to rotate with angular velocity ω about its axis, whilst the outer one remains fixed, we must have

$$\psi_1 = 0 \quad \text{when} \quad \eta = \alpha,$$
$$\psi_1 = \tfrac{1}{2}\omega r^2 \quad \text{when} \quad \eta = \beta.$$

Assume

$$\psi_1 = \sum_2^\infty A_n\frac{\sinh n(\eta-\alpha)}{\sinh n(\beta-\alpha)}\cos n\xi;$$

therefore, when $\eta = \beta$,

$$2\sin\xi \sum_2^\infty A_n \cos n\xi = 4c^2\omega \sum_1^\infty (-1)^{n-1}\varepsilon^{-n\beta}\sin n\xi,$$

now $\quad 2\sin\xi\cos n\xi = \sin(n+1)\,\xi - \sin(n-1)\,\xi\,;$

therefore $\quad -A_2\sin\xi - A_3\sin 2\xi + \quad (A_{n-1} - A_{n+1})\sin n\xi$

$$= 4c^2\omega\{\varepsilon^{-\beta}\sin\xi - \varepsilon^{-2\beta}\sin 2\xi + \quad (-1)^{n-1}\sin n\xi\}\,;$$

therefore $\qquad A_{n+2} - A_n = -\,4c^2\omega\,(1-)^n\varepsilon^{-n\beta}\,;$

therefore $\quad A_n = B + C(-1)^n - \dfrac{4c^2\omega\,(-1)^n\,\varepsilon^{-(n+1)\beta}}{\varepsilon^{-2\beta} - 1}.$

Now $\qquad A_2 = -\,4c^2\omega\,\varepsilon^{-\beta},$

$$A_3 = \quad 4c^2\omega\,\varepsilon^{-2\beta}\,;$$

therefore $\qquad B = -\dfrac{2c^2\omega\,\varepsilon^{-\beta}}{1+\varepsilon^{-\beta}},$

$$C = -\frac{2c^2\omega\,\varepsilon^{-\beta}}{1-\varepsilon^{-2\beta}}(1+\varepsilon^{-\beta})\,;$$

therefore

$$A_n = -\frac{2c^2\omega\,\varepsilon^{-\beta}}{1+\varepsilon^{-\beta}} - \frac{2c^2\omega\,\varepsilon^{-\beta}\,(-1)^n}{1-\varepsilon^{-2\beta}}(1+\varepsilon^{-\beta}-2\varepsilon^{-n\beta})\,;$$

therefore

$$\psi_1 = -\frac{2c^2\omega\,\varepsilon^{-\beta}}{1+\varepsilon^{-\beta}}\sum_2^\infty\left\{1+\frac{(-1)^n}{1-\varepsilon^{-\beta}}(1+\varepsilon^{-\beta}-2\varepsilon^{-n\beta})\right\} \times \frac{\sinh n(\eta-\alpha)}{\sinh n(\beta-\alpha)}\cos n\xi,$$

and

$$\phi_1 = -\frac{2c^2\omega\,\varepsilon^{-\beta}}{1+\varepsilon^{-\beta}}\sum_2^\infty\left\{1+\frac{(-1)^n}{1-\varepsilon^{-\beta}}(1+\varepsilon^{-\beta}-2\varepsilon^{-n\beta})\right\} \times \frac{\sinh n(\eta-\alpha)}{\sinh n(\beta-\alpha)}\sin n\xi.$$

If the outer cylinder had begun to rotate with angular velocity Ω in the same direction, we should have

$$\psi_2 = \tfrac{1}{2}\Omega r^2 \quad \text{when} \quad \eta = \alpha,$$
$$\psi_2 = 0 \qquad \text{when} \quad \eta = \beta.$$

Hence, the values of ψ_2 and ϕ_2 can be obtained from those of ψ_1 and ϕ_1, by changing β into α in the coefficient, and by writing $\sinh n(\beta-\eta)$ and $-\cosh n(\beta-\eta)$ for $\sinh n(\eta-\alpha)$ and $\cosh n(\eta-\alpha)$ respectively.

If both the cylinders had begun to rotate with angular velocities ω and Ω respectively, the current and velocity functions of the initial motion would be $\psi_1+\psi_2$ and $\phi_1+\phi_2$ respectively, and in the particular case in which $\omega=\Omega$ these expressions will hold always for the motion.

The motion of a liquid contained within the cylinder $\eta=\alpha$, when the cylinder is rotating about its axis, may either be investigated independently, or deduced from the preceding case by making $\beta=\infty$.

Putting $\beta=\infty$ in the expression just obtained, we find

$$\psi=-\frac{2c^2\Omega\varepsilon^{-\alpha}}{1+\varepsilon^{-\alpha}}\sum_2^\infty\left\{1+\frac{(-1)^n}{1-\varepsilon^{-\alpha}}(1+\varepsilon^{-\alpha}-2\varepsilon^{-n\alpha})\right\}\varepsilon^{-n(\eta-\alpha)}\cos n\xi$$

$$=-\frac{c^2\Omega}{2\sinh\alpha}\left[\frac{(1-\varepsilon^{-\alpha})\,\varepsilon^{-(\eta-\alpha)}\{\cos 2\xi-\varepsilon^{-(\eta-\alpha)}\cos\xi\}}{\cosh(\eta-\alpha)-\cos\xi}\right.$$

$$\left.+\frac{(1+\varepsilon^{-\alpha})\,\varepsilon^{-(\eta-\alpha)}\{\cos 2\xi+\varepsilon^{-(\eta-\alpha)}\cos\xi\}}{\cosh(\eta-\alpha)+\cos\xi}\right.$$

$$\left.-\frac{2\varepsilon^{-\eta}(\cos 2\xi+\varepsilon^{-\eta}\cos\xi)}{\cosh\eta+\cos\xi}\right].$$

The sum of the first two terms

$$=\frac{\left\{\begin{array}{r}2\varepsilon^{-(\eta-\alpha)}\{\cosh(\eta-\alpha)\cos 2\xi-\varepsilon^{-(\eta-\alpha)}\cos 2\xi\}\\+2\varepsilon^{-\eta}\{\varepsilon^{-(\eta-\alpha)}\cosh(\eta-\alpha)-\cos 2\xi\}\cos\xi\end{array}\right\}}{\frac{1}{2}\{\cosh 2(\eta-\alpha)-\cos 2\xi\}}$$

$$=\frac{\cos 2\xi-\varepsilon^{-2(\eta-\alpha)}+\varepsilon^{-\eta}\{\varepsilon^{-2(\eta-\alpha)}\cos\xi-\cos 3\xi\}}{\frac{1}{2}\{\cosh 2(\eta-\alpha)-\cos 2\xi\}}$$

$$=2.\frac{\varepsilon^{-\eta}\{\varepsilon^{-2(\eta-\alpha)}\cos\xi-\cos 3\xi\}+\sinh 2(\eta-\alpha)}{\cosh 2(\eta-\alpha)-\cos 2\xi}-2;$$

therefore

$$\psi=\frac{c^2\Omega}{\sinh\alpha}\left[1-\frac{\varepsilon^{-\eta}\{\varepsilon^{-2(\eta-\alpha)}\cos\xi-\cos 3\xi\}+\sinh 2(\eta-\alpha)}{\cosh 2(\eta-\alpha)-\cos 2\xi}\right.$$

$$\left.+\frac{\varepsilon^{-\eta}(\cos 2\xi+\varepsilon^{-\eta}\cos\xi)}{\cosh\eta+\cos\xi}\right],$$

and

$$\phi = \frac{c^2\Omega}{\sinh\alpha}\left[\frac{\sin 2\xi + \epsilon^{-\eta}\{\epsilon^{-2(\eta-\alpha)}\sin\xi - \sin 3\xi\}}{\cosh 2(\eta-\alpha) - \cos 2\xi} - \frac{\epsilon^{-\eta}(\sin 2\xi + \epsilon^{-\eta}\sin\xi)}{\cosh\eta + \cos\xi}\right].$$

This result can be easily verified, for putting

$\eta = \alpha$, then

$$\psi = \frac{c^2\Omega}{\sinh\alpha}\left\{1 - 2\epsilon^{-\alpha}\cos\xi + \frac{\epsilon^{-\alpha}(\cos 2\xi + \epsilon^{-\alpha}\cos\xi)}{\cosh\alpha + \cos\xi}\right\}$$

$$= \frac{c^2\Omega}{\cosh\alpha + \cos\xi}$$

$$= \tfrac{1}{2}\Omega r^2.$$

We shall, in the next place, investigate the motion of a vortex within the cylinder $\eta = \beta$.

Let R be the distance between two points ξ, η; and ξ', η' within the cylinders, and let

$$\psi = \log R + \Psi,$$

where Ψ is a function of ξ and η satisfying the equation $\nabla^2\Psi = 0$, such that at every point within the cylinder Ψ and its derivatives of the first order are finite and continuous. Then if Ψ be determined so that $\psi = 0$ when $\eta = \beta$, ψ is the current function of the motion due to a vortex at the point ξ', η'. We also see that Ψ is the potential within the cylinder of the electricity induced thereon, by an infinite electrified line which is parallel to the axis and is situated at the point $\xi'\eta'$.

Let

$$\Psi = \sum_1^\infty \frac{\epsilon^{-n(\eta-\beta)}}{n}(A_n\cos n\xi + B_n\sin n\xi) - \eta' + \beta - \log c\sqrt{2}$$
$$+ \tfrac{1}{2}\log(\cosh 2\eta' + \cos 2\xi').$$

Since $\eta' > \beta$, the value of $\log R$ at the surface is

$$\log c\sqrt{2} + \eta' - \beta - \tfrac{1}{2}\log(\cosh 2\eta' + \cos 2\xi')$$
$$- \sum_1^\infty \frac{1}{n}\{\epsilon^{-n(\eta'+\beta)}\cos n(\xi'+\xi) + \epsilon^{-n(\eta'-\beta)}\cos n(\xi'-\xi)$$
$$- (-1)\epsilon^{-2n\beta}\cos 2n\xi\};$$

therefore

$$\sum_1^\infty \frac{1}{n}(A_n \cos n\xi + \beta_n \sin n\xi) - \sum_1^\infty \frac{1}{n}\{\varepsilon^{-n(\eta'+\beta)} \cos n(\xi'+\xi) + \varepsilon^{-n(\eta'-\beta)} \cos n(\xi'-\xi) - (-1)^n \varepsilon^{-2n\beta} \cos 2n\xi\} = 0;$$

therefore

$$B_n = -\{\varepsilon^{-n(\eta'+\beta)} - \varepsilon^{-n(\eta'-\beta)}\} \sin n\xi',$$

$$A_{2n+1} = \{\varepsilon^{-(2n+1)(\eta'+\beta)} + \varepsilon^{-(2n+1)(\eta'-\beta)}\} \cos(2n+1)\xi',$$

$$A_{2n} = \{\varepsilon^{-2n(\eta'+\beta)} + \varepsilon^{-2n(\eta'-\beta)}\} \cos 2n\xi' - (-1)^n \varepsilon^{-2n\beta};$$

therefore

$$\sum \frac{\varepsilon^{-n(\eta-\beta)}}{n}(A_n \cos n\xi + \beta_n \sin n\xi)$$

$$= \sum \frac{1}{n}\{\varepsilon^{-n(\eta'+\eta)} \cos n(\xi'+\xi) + \varepsilon^{-n(\eta'+\eta-2\beta)} \cos n(\xi'-\xi) - (-1)^n . \varepsilon^{-2n\eta} \cos 2n\xi\}$$

$$= \tfrac{1}{2} \log \frac{\varepsilon^{\eta'+\eta}}{2\{\cosh(\eta'+\eta) - \cos(\xi'+\xi)\}} + \tfrac{1}{2} \log \frac{\varepsilon^{\eta'+\eta-2\beta}}{2\{\cosh(\eta'+\eta-2\beta) - \cos(\xi'-\xi)\}} + \tfrac{1}{2} \log(\cosh 2\eta + \cos 2\xi) + \tfrac{1}{2} \log 2 - \eta$$

$$= -\tfrac{1}{2} \log 2 + \eta' - \beta + \tfrac{1}{2} \log \frac{\cosh 2\eta + \cos 2\xi}{\{\cosh(\eta'+\eta) - \cos(\xi'+\xi)\}\{\cosh(\eta'+\eta-2\beta) - \cos(\xi'-\xi)\}};$$

therefore $\Psi = -\log 2c + \tfrac{1}{2} \log \frac{(\cosh 2\eta + \cos 2\xi)(\cosh 2\eta' + \cos 2\xi')}{\{\cosh(\eta'+\eta) - \cos(\xi'+\xi)\}\{\cosh(\eta'+\eta-2\beta) - \cos(\xi'-\xi)\}},$

and $$\psi = \tfrac{1}{2} \log \frac{\cosh(\eta'-\eta) - \cos(\xi'-\xi)}{\cosh(\eta'+\eta-2\beta) - \cos(\xi'-\xi)}.$$

This expression is of the same form as the current function due to a vortex external to an elliptic cylinder, and the discussion of the images can be carried on in the same

manner as has been done by Mr. Hicks in the case of a vortex external to an elliptic cylinder.*

The expression for ψ which we have just obtained enables us to give a physical meaning to the functions ξ and η, for if we put $\eta' = \infty$ in this expression, ψ will be the potential of the electric field within the cylinder, when the cylinder is under the action of an electrified line situated at the centre. We thus obtain

$$\psi = -(\eta - \beta).$$

Hence the curves $\eta = \text{const.}$ are the equi-potential curves, and the curves $\xi = \text{const.}$ are the lines of force.

The surface density

$$\sigma = \frac{1}{4\pi}\left(J\frac{d\psi}{d\eta}\right)_{\eta=\beta}$$

$$= -\frac{c^2 p}{2\pi r^4 \sinh 2\beta} \propto \frac{p}{r^4}.$$

The circumstance that the current function due to a vortex within the cylinder $\eta = \beta$ is of the same form as that due to a vortex outside an elliptic cylinder, suggests the enquiry whether a similar proposition is true with respect to a vortex within an elliptic cylinder. The current function due to such a vortex has been obtained by Mr. C. V. Coates,† who finds the expression

$$\psi = \log \frac{\{\mathrm{sn}^2\alpha(\xi+\xi') - \mathrm{sn}^2\alpha\iota(\eta+\eta')\}\{\mathrm{sn}^2\alpha(\xi-\xi') - \mathrm{sn}^2\alpha\iota(\eta-\eta')\}}{\{1-k^2\mathrm{sn}^2\alpha(\xi+\xi')\mathrm{sn}^2\alpha\iota(\eta-\eta')\}\{1-k^2\mathrm{sn}^2\alpha(\xi-\xi')\mathrm{sn}^2\alpha\iota(\eta+\eta')\}},$$

where

$$\xi + \iota\eta = \cos^{-1}\frac{x+\iota y}{c},$$

$$\alpha = \frac{K}{\pi} = \frac{K'}{2\beta};$$

and $\eta = \beta$ is the transverse section of the elliptic cylinder.

We shall examine this function and shall show that if $\xi + \iota\eta$ be understood to mean $\sec^{-1}\dfrac{x+\iota y}{c}$, ψ satisfies all the conditions necessary to be satisfied by the current function due to a vortex at a point ξ', η' external to the cylinder $\eta = \beta$.

---

* *Quarterly Journal of Mathematics*, vol. XVII. p. 327.

† *Quarterly Journal of Mathematics*, vol. XVI. p. 81.

ψ obviously vanishes when $\eta = \beta$ by virtue of the relation

$$k \operatorname{sn}(u + \iota K') \operatorname{sn} u = 1.$$

Also

$$\frac{d\psi}{d\xi} = 2\alpha \operatorname{sn}\alpha(\xi+\xi') \operatorname{cn}\alpha(\xi+\xi') \operatorname{dn}\alpha(\xi+\xi') \times \left\{ \frac{1}{\operatorname{sn}^2\alpha(\xi+\xi') - \operatorname{sn}^2\alpha\iota(\eta+\eta')} + \frac{k^2 \operatorname{sn}^2\alpha\iota(\eta-\eta')}{1-k^2\operatorname{sn}^2\alpha(\xi+\xi')\operatorname{sn}^2\alpha\iota(\eta-\eta')} \right\}$$

$$+ 2\alpha \operatorname{sn}\alpha(\xi-\xi') \operatorname{cn}\alpha(\xi-\xi') \operatorname{dn}\alpha(\xi-\xi') \times \left\{ \frac{1}{\operatorname{sn}^2\alpha(\xi-\xi') - \operatorname{sn}^2\alpha\iota(\eta-\eta')} + \frac{k^2 \operatorname{sn}^2\alpha\iota(\eta+\eta')}{1-k^2\operatorname{sn}^2\alpha(\xi-\xi')\operatorname{sn}^2\alpha\iota(\eta-\eta')} \right\},$$

$$\frac{d\psi}{d\eta} = -2\alpha\iota \operatorname{sn}\alpha\iota(\eta+\eta') \operatorname{cn}\alpha\iota(\eta+\eta') \operatorname{dn}\alpha\iota(\eta+\eta') \times \left\{ \frac{1}{\operatorname{sn}^2\alpha(\xi+\xi') - \operatorname{sn}^2\alpha\iota(\eta+\eta')} - \frac{k^2 \operatorname{sn}^2\alpha(\xi-\xi')}{1-k^2\operatorname{sn}^2\alpha(\xi-\xi')\operatorname{sn}^2\alpha\iota(\eta+\eta')} \right\}$$

$$- 2\alpha\iota \operatorname{sn}\alpha\iota(\eta+\eta') \operatorname{cn}\alpha\iota(\eta-\eta') \operatorname{dn}\alpha\iota(\eta-\eta') \times \left\{ \frac{1}{\operatorname{sn}^2\alpha(\xi-\xi') - \operatorname{sn}^2\alpha\iota(\eta-\eta')} - \frac{k^2 \operatorname{sn}^2\alpha(\xi+\xi')}{1-k^2\operatorname{sn}^2\alpha(\xi+\xi')\operatorname{sn}^2\alpha\iota(\eta-\eta')} \right\}.$$

When $\eta = 0$, these expressions change sign when ξ is changed into $2\pi - \xi$ (or $-\xi$ which is the same thing), so that there is no discontinuity along Fx.

Also $\frac{d\psi}{d\xi}$ and $\frac{d\psi}{d\eta}$ are each of the form

$$F(\xi'+\xi, \eta'+\eta) - F(\xi'-\xi, \eta'-\eta) + f(\xi'+\xi, \eta'-\eta) - f(\xi'-\xi, \eta'+\eta),$$

the value of which at a point $\delta\xi$, $\delta\eta$ in the neighbourhood of the foci is

$$2\left\{ \left(\frac{dF}{d\xi'} + \frac{df}{d\xi'} \right) \delta\xi + \left(\frac{dF}{d\eta'} - \frac{df}{d\eta'} \right) \delta\eta \right\},$$

the value of J at this point is $\frac{1}{c\sqrt{(\delta\xi^2 + \delta\eta^2)}}$, and as the derivatives of F and f with respect to ξ' and η' do not become infinite at the foci, the velocities are finite at these points.

It remains to show that the velocities vanish at infinity. At a great distance from the origin $\xi = \frac{1}{2}\pi - \delta\xi$, $\eta = \delta\eta$ and the value of J at this point is $\frac{1}{c}(\delta\xi^2 + \delta\eta^2)$, and the values of

$\frac{d\psi}{d\eta}$ and $\frac{d\psi}{d\xi}$ are not infinite at this point, so that the velocities at a great distance from the origin are infinitesimal quantities of the second order.

Before proceeding to discuss the motion of the cylinder $\eta=\beta$ in an infinite liquid, we shall require the following preliminary proposition, viz. that the motion of a liquid parallel to the axis if y can be obtained by placing two equal vortices of opposite circulations on the axis of x on either side of the origin and at equal distances from it, and making them move off to infinity whilst their strengths increase indefinitely, but so that the velocity remains finite.

Thus if ξ, $-\xi$ be the distances of the vortices from the origin; m, $-m$ their strengths, the current function at x, y is

$$\begin{aligned}\psi &= \tfrac{1}{2}m\log\frac{(x-\xi)^2+y^2}{(x+\xi)^2+y^2}\\ &= \tfrac{1}{2}m\log\frac{1-\dfrac{2x}{\xi}+\dfrac{x^2+y^2}{\xi^2}}{1+\dfrac{2x}{\xi}+\dfrac{x^2+y^2}{\xi^2}}\\ &= -\frac{2m}{\xi}x\\ &= -Vx.\end{aligned}$$

Hence ψ is the current function of a liquid moving parallel to y with velocity $-V$.

We shall employ this method to find the motion of a liquid which is flowing past the cylinder $\eta=\beta$, the velocity of the liquid at infinity being parallel to the axis of y and equal to $-V$.

The current function due to a vortex at $\eta=0$, $\xi=\frac{1}{2}\pi-\delta\xi$ is

$$m\log\frac{\left\{\mathrm{sn}^2\frac{K}{\pi}(\xi+\tfrac{1}{2}\pi-\delta\xi)-\mathrm{sn}^2\frac{\iota K'\eta}{2\beta}\right\}\left\{\mathrm{sn}^2\frac{K}{\pi}(\xi-\tfrac{1}{2}\pi+\delta\xi)-\mathrm{sn}^2\frac{\iota K'\eta}{2\beta}\right\}}{\left\{1-k^2\mathrm{sn}^2\frac{K}{\pi}(\xi+\tfrac{1}{2}\pi-\delta\xi)\mathrm{sn}^2\frac{\iota K'\eta}{2\beta}\right\}\left\{1-k^2\mathrm{sn}^2\frac{K}{\pi}(\xi-\tfrac{1}{2}\pi+\delta\xi)\mathrm{sn}^2\frac{\iota K'\eta}{2\beta}\right\}}.$$

Putting

$$\frac{K\xi}{\pi}+\frac{\iota K'\eta}{2\beta}+\tfrac{1}{2}K=u,$$

$$\frac{K\xi}{\pi}-\frac{\iota K'\eta}{2\beta}+\tfrac{1}{2}K=v,$$

$$\frac{K\delta\xi}{\pi}=\delta u.$$

This becomes

$$m \log \operatorname{sn}(u-\delta u)\operatorname{sn}(v-\delta u)\operatorname{sn}(u-K+\delta u)\operatorname{sn}(v-K+\delta u).$$

The current function due to the opposite vortex at $\eta = 0$, $\xi = \frac{3}{2}\pi - \delta\xi$ is

$$-m \log \operatorname{sn}(u+K-\delta u)\operatorname{sn}(v+K-\delta u)\operatorname{sn}(u-2K+\delta u)\operatorname{sn}(v-2K+\delta u),$$

Remembering that $\operatorname{sn}(u \pm 2K) = -\operatorname{sn} u$, the current function for the whole motion may be written

$$\psi_1 = m \log \frac{\operatorname{sn}(u-\delta u)\operatorname{sn}(v-\delta u)\operatorname{sn}(u+K+\delta u)\operatorname{sn}(v+K+\delta u)}{\operatorname{sn}(u+\delta u)\operatorname{sn}(v+\delta u)\operatorname{sn}(u+K-\delta u)\operatorname{sn}(v+K+\delta u)}.$$

Now

$$\log \operatorname{sn}(u+\delta u) = \log \operatorname{sn} u + \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}.\delta u;$$

and, therefore,

$$\psi_1 = -2m\delta u \left\{ \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} + \frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} - \frac{\operatorname{cn}(u+K)\operatorname{dn}(u+K)}{\operatorname{sn}(u+K)} - \frac{\operatorname{cn}(v+K)\operatorname{dn}(v+K)}{\operatorname{sn}(v+K)} \right\}$$

$$= -2m\delta u \left\{ \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u + k'^2 \operatorname{sn}^2 u}{\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u} + \frac{\operatorname{cn}^2 v \operatorname{dn}^2 v + k'^2 \operatorname{sn}^2 v}{\operatorname{sn} v \operatorname{cn} v \operatorname{dn} v} \right\}$$

$$= -4m\delta u \left\{ \frac{\operatorname{dn} 2u}{\operatorname{sn} 2u} + \frac{\operatorname{dn} 2v}{\operatorname{sn} 2v} \right\}$$

$$= -M \left\{ \frac{1}{\operatorname{cn}\left(\frac{2K\xi}{\pi} + \frac{\iota K'\eta}{\beta}\right)} + \frac{1}{\operatorname{cn}\left(\frac{2K\xi}{\pi} - \frac{\iota K'\eta}{\beta}\right)} \right\}.$$

We shall now verify this result ; ψ obviously vanishes when $\eta = \beta$, also putting $\gamma = \frac{2K}{\pi} = \frac{K'}{\beta}$,

$$\frac{d\psi_1}{d\eta} = -M\iota\gamma \left\{ \frac{\operatorname{sn}\gamma(\xi+\iota\eta)\operatorname{dn}\gamma(\xi+\iota\eta)}{\operatorname{cn}^2\gamma(\xi+\iota\eta)} - \frac{\operatorname{sn}\gamma(\xi-\iota\eta)\operatorname{dn}\gamma(\xi-\iota\eta)}{\operatorname{cn}^2\gamma(\xi-\iota\eta)} \right\},$$

$$\frac{d\psi_1}{d\xi} = -M\gamma \left\{ \frac{\operatorname{sn}\gamma(\xi+\iota\eta)\operatorname{dn}\gamma(\quad)}{\operatorname{cn}^2\gamma(\quad)} + \frac{\operatorname{sn}\gamma(\xi-\iota\eta)\operatorname{dn}\gamma(\quad)}{\operatorname{cn}^2\gamma(\quad)} \right\}.$$

As we pass from the positive to the negative side of the line Fx, $\frac{d\psi_1}{d\xi}$ changes sign, and $\frac{d\psi_1}{d\eta}$ vanishes when $\eta = 0$, so there is no discontinuity along this line; also at the foci these quantities are infinitesimals of the first order, and as $J = \frac{1}{c\sqrt{(\delta\eta^2+\delta\xi^2)}}$ at these points, the velocities at the foci are finite.

At a great distance from the origin $\eta = \delta\eta$, $\xi = \frac{1}{2}\pi - \delta\xi$, so that this point

$$\frac{d\psi_1}{d\eta} = -\frac{M\iota\gamma}{k'}\left(\frac{1}{\gamma^2(\delta\xi - \iota\delta\eta)^2} - \frac{1}{\gamma^2(\delta\xi + \iota\delta\eta)^2}\right)$$

$$= \frac{4M}{k'\gamma} \cdot \frac{\delta\xi\,\delta\eta}{(\delta\xi^2 + \delta\eta^2)^2}.$$

But $$\frac{y}{x} = \tan\theta = \tanh\eta \tan\xi = \frac{\delta\eta}{\delta\xi},$$

and $$J = \frac{1}{c}(\delta\xi^2 + \delta\eta^2);$$

therefore $$J\frac{d\psi_1}{d\eta} = \frac{2M}{k'c\gamma} \cdot \sin 2\theta = -q,$$

and $$J\frac{d\psi_1}{d\xi} = -\frac{2M}{k'c\gamma} \cdot \cos 2\theta = p.$$

If l, m be the direction cosines of the normal at $\delta\eta$, $\frac{1}{2}\pi - \delta\xi$

$$l = \frac{(2r^2 - a^2)\,x}{(a^4x^2 + b^4y^2)^{\frac{1}{2}}} = \frac{\cos\theta\left(\frac{a^2}{b^2}\cos 2\theta + 2\sin^2\theta\right)}{\left(\frac{a^4}{b^4}\cos^2\theta + \sin^2\theta\right)^{\frac{1}{2}}}$$

$$= \sin 2\theta$$

$$m = \frac{(2r^2 - b^2)\,y}{(a^4x^2 + b^4y^2)^{\frac{1}{2}}} = \frac{\sin\theta\left(\frac{2a^2}{b^2}\cos^2\theta - \cos 2\theta\right)}{\left(\frac{a^4}{b^4}\cos^2\theta + \sin^2\theta\right)^{\frac{1}{2}}}$$

$$= -\cos 2\theta;$$

therefore velocity parallel to x,

$$= -pl - qm$$

$$= \frac{2M}{k'c\gamma}(\sin 2\theta \cos 2\theta - \cos 2\theta \sin 2\theta)$$

$$= 0,$$

velocity parallel to y,

$$= ql - pm = -\frac{2M}{k'c\gamma} = -V \qquad \text{(fig. 18)};$$

therefore

$$\psi_1 = -\tfrac{1}{2}\, Vck'\gamma \left\{ \frac{1}{\operatorname{cn}\gamma(\xi + \iota\eta)} + \frac{1}{\operatorname{cn}\gamma(\xi + \iota\eta)} \right\}.$$

Hence if the cylinder be moving with velocity V parallel to the axis of y, and χ_1 be current function,

$$\chi_1 = \psi_1 + Vx,$$

and the velocity function is

$$\phi_1 = \tfrac{1}{2}\, Vc\gamma\, k'\iota \left\{ \frac{1}{\operatorname{cn}\gamma(\xi + \iota\eta)} - \frac{1}{\operatorname{cn}\gamma(\xi - \iota\eta)} \right\} + Vy.$$

For the corresponding case of motion parallel to x, we must place two vortices on the axis of y at the points $\xi' = \frac{1}{2}\pi$, $\eta' = \delta\eta$; $\xi' = \frac{3}{2}\pi$, $\eta' = \delta\eta$, the current function for the first vortex is

$$m \log \frac{\left\{ \operatorname{sn}^2\left(\frac{K\xi}{\pi} + \tfrac{1}{2}K\right) - \operatorname{sn}^2 \frac{\iota K'}{2\beta}(\eta + \delta\eta) \right\} \left\{ \operatorname{sn}^2\left(\frac{K\xi}{\pi} - \tfrac{1}{2}K\right) - \operatorname{sn}^2 \frac{\iota K'}{2\beta}(\eta - \delta\eta) \right\}}{\left\{ 1 - k^2 \operatorname{sn}^2\left(\frac{K\xi}{\pi} + \tfrac{1}{2}K\right) \operatorname{sn}^2 \frac{\iota K'}{2\beta}(\eta - \delta\eta) \right\} \left\{ 1 - k^2 \operatorname{sn}^2\left(\frac{K\xi}{\pi} - \tfrac{1}{2}K\right) \operatorname{sn}^2 \frac{\iota K'}{2\beta}(\eta + \delta\eta) \right\}}.$$

Putting for a moment

$$\frac{K\xi}{\pi} + \tfrac{1}{2}K = x,$$

$$\frac{\iota K'\eta}{2\beta} = y,$$

$$\frac{\iota K'\delta\eta}{2\beta} = \delta y;$$

and remembering that $\operatorname{sn}(x-K) = -\dfrac{\operatorname{cn} x}{\operatorname{dn} x}$, this becomes

$$m \log \frac{\{\operatorname{sn}^2 x - \operatorname{sn}^2(y+\delta y)\}\quad \{\operatorname{cn}^2 x - \operatorname{dn}^2 x \operatorname{sn}^2(y-\delta y)\}}{\{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2(y-\delta y)\}\ \{\operatorname{dn}^2 x - k^2 \operatorname{cn}^2 x \operatorname{sn}^2(y+\delta y)\}}.$$

Now

$$\frac{\operatorname{cn}^2 x - \operatorname{dn}^2 x \operatorname{sn}^2(y-\delta y)}{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2(y-\delta y)} = \operatorname{cn}(x+y-\delta y)\operatorname{cn}(x-y+\delta y),$$

and

$$\frac{\operatorname{dn}^2 x - k^2 \operatorname{cn}^2 x \operatorname{sn}^2(y+\delta y)}{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2(y+\delta y)} = \operatorname{dn}(x+y-\delta y)\operatorname{dn}(x-y-\delta y);$$

therefore the current function

$$= m \log \frac{\operatorname{sn}(u+\delta u)\operatorname{sn}(v-\delta u)\operatorname{cn}(u-\delta u)\operatorname{cn}(v+\delta u)}{\operatorname{dn}(u+\delta u)\operatorname{dn}(v-\delta u)}$$

$$= m \log \frac{1}{k'^2} \operatorname{cn}(u+K+\delta u)\operatorname{cn}(v+K-\delta u)\operatorname{cn}(u-\delta u)\operatorname{cn}(v+\delta u),$$

when u, v and δu have the same meanings as before.

The value of the current function for the other vortex is

$$-m \log \frac{1}{k'^2} \operatorname{cn}(u+\delta u)\operatorname{cn}(v-\delta u)\operatorname{cn}(u+K-\delta u)\operatorname{cn}(v+K+\delta u);$$

therefore the current function for the whole motion is

$$\psi_2 = m \log \frac{\operatorname{cn}(u-\delta u)\operatorname{cn}(v+\delta u)\operatorname{cn}(u+K+\delta u)\operatorname{cn}(v+K-\delta u)}{\operatorname{cn}(u+\delta u)\operatorname{cn}(v-\delta u)\operatorname{cn}(u+K-\delta u)\operatorname{cn}(v+K+\delta u)}.$$

Now $\log \operatorname{cn}(u-\delta u) = \log \operatorname{cn} u + \dfrac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}.\delta u;$

therefore

$$\psi_2 = 2m\,\delta u \left\{ \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} - \frac{\operatorname{sn} v \operatorname{dn} v}{\operatorname{cn} v} - \frac{\operatorname{sn}(u+K)\operatorname{dn}(u+K)}{\operatorname{cn}(u+K)} + \frac{\operatorname{sn}(v+K)\operatorname{dn}(v+K)}{\operatorname{cn}(v+K)} \right\}$$

$$= 2m\,\delta u \left\{ \frac{1-k^2 \operatorname{sn}^4 u}{\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u} - \frac{1-k^2 \operatorname{sn}^4 v}{\operatorname{sn} v \operatorname{cn} v \operatorname{dn} v} \right\}$$

$$= M \left\{ \frac{1}{\operatorname{sn} 2u} - \frac{1}{\operatorname{sn} 2v} \right\},$$

$$= M\left\{\frac{1}{\mathrm{sn}\left(\frac{2K\xi}{\pi}+\frac{\iota K'\eta}{\beta}+K\right)}-\frac{1}{\mathrm{sn}\left(\frac{2K\xi}{\pi}-\frac{\iota K'\eta}{\beta}+K\right)}\right\}$$

$$= M\left\{\frac{\mathrm{dn}\,\gamma(\xi+\iota\eta)}{\mathrm{cn}\,\gamma(\xi+\iota\eta)}-\frac{\mathrm{dn}\,\gamma(\xi-\iota\eta)}{\mathrm{cn}\,\gamma(\xi-\iota\eta)}\right\},$$

which evidently vanishes when $\eta=\beta$. Also it can be shown that the velocities at the foci are finite and that there is no discontinuity along the line Fx; it remains therefore to determine M,

$$\frac{d\psi_2}{d\eta}=-M\iota\gamma\left\{\frac{\mathrm{cn}2u\,\mathrm{dn}2u}{\mathrm{sn}^2 2u}+\frac{\mathrm{cn}2v\,\mathrm{dn}2v}{\mathrm{sn}^2 2v}\right\},$$

and putting $\xi=\frac{1}{2}\pi-\delta\xi$, $\eta=\delta\eta$,

$$\frac{d\psi_2}{d\eta}=\frac{M\iota}{\gamma}\left\{\frac{1}{(\delta\xi-\iota\delta\eta)^2}+\frac{1}{(\delta\xi+\iota\delta\eta)^2}\right\}$$

$$=\frac{2M\iota}{\gamma}\cdot\frac{\delta\xi^2-\delta\eta^2}{(\delta\xi^2+\delta\eta^2)^2};$$

therefore

$$J\frac{d\psi_2}{d\eta}=\frac{2M\iota}{\gamma c}\cdot\cos 2\theta=-q,$$

and

$$J\frac{d\psi_2}{d\xi}=\frac{2M\iota}{\gamma c}\cdot\sin 2\theta=\quad p;$$

therefore velocity parallel to x,

$$=-\frac{2M\iota}{\gamma c}=-U,$$

velocity parallel to $y=0$.

Therefore

$$\psi_2=\frac{U\gamma c}{2\iota}\left\{\frac{1}{\mathrm{sn}\{K+\gamma(\xi+\iota\eta)\}}-\frac{1}{\mathrm{sn}\{K+\gamma(\xi-\iota\eta)\}}\right\}.$$

Hence if the cylinder be moving with velocity U parallel to the axis of x, the current function is

$$\chi_2 = \psi_2 - Uy,$$

and the velocity function is

$$\phi_2 = -\frac{U\gamma c}{2}\left\{\frac{1}{\mathrm{sn}\{K+\gamma(\xi+\iota\eta)\}}+\frac{1}{\mathrm{sn}\{K+\gamma(\xi-\iota\eta)\}}\right\}+Ux.$$

The function ψ is the potential of the electric field when the cylinder is under the action of an electrified line at the point $\xi'\eta'$. The effect of putting $\xi'=\frac{1}{2}\pi$, $\eta'=0$ is to make the line move off to infinity, and we thus obtain for the potential of an electrified cylinder under the action of no electric forces the expression

$$V = \log \mathrm{sn}\,u\,\mathrm{sn}\,(u-K)\,\mathrm{sn}\,v\,\mathrm{sn}\,(v-K) \ldots\ldots\ldots (1),$$

where u and v have the same meanings as before. This may be put into the form

$$V = \log \frac{1}{k^2}\frac{\mathrm{dn}\left(\frac{2K\xi}{\pi}, k\right)\mathrm{dn}\left(\frac{K'\eta}{\beta}, k'\right) - k'\,\mathrm{cn}\left(\frac{K'\eta}{\beta}, k'\right)}{\mathrm{dn}\left(\frac{2K\xi}{\pi}, k\right)\mathrm{dn}\left(\frac{K'\eta}{\beta}, k'\right) + k'\,\mathrm{cn}\left(\frac{K'\eta}{\beta}, k'\right)},$$

The lines of force are given by the equation

$$\epsilon^{\iota\phi} = \frac{\mathrm{sn}\,u\,\mathrm{sn}\,(u-K)}{\mathrm{sn}\,v\,\mathrm{sn}\,(v-K)};$$

therefore
$$\cot\tfrac{1}{2}\phi = \iota\frac{\mathrm{sn}\,(u+v)\,\mathrm{cn}\,(u-v)}{\mathrm{cn}\,(u+v)\,\mathrm{cn}\,(u-v)}$$

$$= -\frac{\mathrm{cn}\frac{2K\xi}{\pi}\,\mathrm{sn}\frac{\iota K'\eta}{\beta}}{k'\,\mathrm{sn}\frac{2K\xi}{\pi}\,\mathrm{cn}\frac{\iota K'\eta}{\beta}};$$

therefore

$$\phi = -2\tan^{-1}\frac{k'\,\mathrm{sn}\left(\frac{2K\xi}{\pi}, k\right)\mathrm{sn}\left(\frac{K'\eta}{\beta}, k'\right)}{\mathrm{cn}\left(\frac{2K\xi}{\pi}, k\right)}.$$

Taking the first form of V we find

$$\frac{dV}{d\eta} = \frac{\iota K'}{2\beta}\left(\frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u - k'^2\,\mathrm{sn}^2 u}{\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u} - \frac{\mathrm{cn}^2 v\,\mathrm{dn}^2 v - k'^2\,\mathrm{sn}^2 v}{\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v}\right)$$

$$= \frac{\iota K'}{\beta}\left(\frac{\mathrm{cn}\,2u}{\mathrm{sn}\,2u} - \frac{\mathrm{cn}\,2v}{\mathrm{sn}\,2v}\right);$$

therefore

$$\left(\frac{dV}{d\eta}\right)_{\eta=\beta} = \frac{2\iota K'}{\beta}\cdot\frac{\mathrm{cn}\left(\dfrac{2K\xi}{\pi} + K + \iota K'\right)}{\mathrm{sn}\left(\dfrac{2K\xi}{\pi} + K + \iota K'\right)}$$

$$= \frac{2Kk'}{\beta}\cdot\frac{1}{\mathrm{dn}\dfrac{2K\xi}{\pi}};$$

therefore the surface density

$$\sigma = \frac{K'k'c^2}{\pi\beta\sinh 2\beta}\cdot\frac{p}{r^4\,\mathrm{dn}\dfrac{2K\xi}{\pi}}.$$

If $\xi + \iota\eta = \cos^{-1}\dfrac{x+\iota y}{c}$ the expressions for V and ϕ give the potential and the lines of force for an elliptic cylinder under the action of an infinite electrified line at the centre. In this case

$$J = \frac{2p}{c^2\sinh 2\beta},$$

so that the surface density

$$\sigma = -\frac{Kk'}{\pi\beta c^2\sinh 2\beta}\cdot\frac{p}{\mathrm{dn}\dfrac{2K\xi}{\pi}}$$

If the cylinder be placed in a field of uniform and parallel electric force, the expressions for ψ_1 and ψ_2, which we have obtained above, enables us to find the electrification. For if the potential of the uniform and parallel field of force be $-Ax - By$, we must find a function V such that $V = 0$,

when $\eta=\beta$ and $-\frac{dV}{dx}=A$, and $-\frac{dV}{dy}=B$ at infinity. The required function is obviously

$$V=-\frac{Ak'c\gamma}{2}\left\{\frac{1}{\operatorname{cn}\gamma(\xi+\iota\eta)}+\frac{1}{\operatorname{cn}\gamma(\xi-\iota\eta)}\right\}$$

$$-\frac{Bc\gamma}{2\iota}\left[\frac{1}{\operatorname{sn}\{K+\gamma(\xi+\iota\eta)\}}-\frac{1}{\operatorname{sn}\{K+\gamma(\xi-\iota\eta)\}}\right],$$

$$\left(\frac{dV}{d\eta}\right)_{\eta=\beta}=kk'c\gamma^2\,\frac{A\operatorname{cn}\frac{2K\xi}{\pi}+Bk'\operatorname{sn}\frac{2K\xi}{\pi}}{\operatorname{dn}^2\frac{2K\xi}{\pi}},$$

and

$$\sigma=\frac{kk'c^3\gamma^2}{2\pi\sinh 2\beta}\cdot\frac{p\left(A\operatorname{cn}\frac{2K\xi}{\pi}+Bk'\operatorname{sn}\frac{2K\xi}{\pi}\right)}{r^4\operatorname{dn}^2\frac{2K\xi}{\pi}},$$

ERRATUM.

p. 191, line 10 from the bottom, for *doubled* read *coincident*.

NOTE ON THE COMPOSITIONS OF A NUMBER AS A SUM OF TWO AND FOUR UNEVEN SQUARES.

By J. W. L. Glaisher.

IF a number is expressible as a sum of four uneven squares it must be of the form $4N$, and if it is expressible as a sum of two uneven squares it must be of the form $2N$, where N denotes an uneven number; and it is known that the number of compositions of $4N$ as a sum of four uneven squares is equal to the sum of the divisors of N, and that the number of compositions of $2N$ as a sum of two uneven squares is equal to the excess of the number of divisors of N of the form $4m+1$ over the number of those of the form $4m+3$.

The object of the present note is to point out a curious relation between the compositions of $4N$ as a sum of four uneven squares and those of $2N$ as a sum of two uneven squares.

The relation is most easily explained by means of an example.

Let $4N=68$, then $N=17$, and the number of compositions is therefore 18. The number 68 can be resolved into a sum of four uneven squares in two ways, viz.

$$1^2+3^2+3^2+7^2,$$
$$3^2+3^2+5^2+5^2;$$

and these give rise to the eighteen compositions

$$1^2+3^2+3^2+7^2,$$
$$1^2+3^2+7^2+3^2,$$
$$1^2+7^2+3^2+3^2,$$
$$3^2+1^2+3^2+7^2,$$
$$3^2+1^2+7^2+3^2,$$
$$3^2+3^2+1^2+7^2,$$
$$3^2+3^2+7^2+1^2,$$
$$3^2+7^2+1^2+3^2,$$
$$3^2+7^2+3^2+1^2,$$
$$7^2+1^2+3^2+3^2,$$
$$7^2+3^2+1^2+3^2,$$
$$7^2+3^2+3^2+1^2,$$

$$3^2+3^2+5^2+5^2,$$
$$3^2+5^2+3^2+5^2,$$
$$3^2+5^2+5^2+3^2,$$
$$5^2+3^2+3^2+5^2,$$
$$5^2+3^2+5^2+3^2,$$
$$5^2+5^2+3^2+3^2.$$

The excess of the number of $4m+1$ divisors of 17 over the number of $4m+3$ divisors $=2$; and $2N$, $=34$, can be expressed as a sum of two uneven squares in only one way, viz. 3^2+5^2. The two compositions are therefore

$$3^2+5^2,$$
$$5^2+3^2.$$

The relation is as follows:

Consider only the squares that occupy the same place in the compositions of $4N$, say the first square in each composition, and write down their square roots with the positive sign when of the form $4m+1$ and with the negative sign when of the form $4m+3$. Following this rule we obtain

$$A = 1+1+1-3-3-3-3-3-3$$

$$-7-7-7-3-3-3+5+5+5.$$

Consider now the compositions of $2N$ and multiply together the square roots of the two squares in each composition, the signs of the square roots being as before taken to be positive or negative according as they are of the forms $4m+1$ or $4m+3$, and add together the products so obtained. We thus find

$$B = -3\times 5+5\times -3.$$

Then the relation is that $A=B$.

In this case

$$A = 3-18-21-9+15 = -30,$$

$$B = -15-15 \qquad\qquad = -30.$$

As another example, take $4N=100$; then $N=25$, and the sum of the divisors of $N=1+5+25=31$.

The resolutions of 100 into sums of four uneven squares are

(i) $1^2+3^2+3^2+9^2$,

(ii) $1^2+5^2+5^2+7^2$,

(iii) $1^2+1^2+7^2+7^2$,

(iv) $5^2+5^2+5^2+5^2$.

The resolution (i) gives rise to twelve compositions, in three of which 1^2 stands first, in six of which 3^2 stands first, and in three of which 9^2 stands first.

(ii) gives rise to twelve compositions: 1^2 stands first in three, 5^2 in six, 7^2 in three.

(iii) gives rise to six compositions: 1^2 stands first in three, and 7^2 in three.

(iv) gives rise to only one composition.

Thus, following the rule, we obtain

from (i), $1 \times 3 - 3 \times 6 + 9 \times 3$,

„ (ii), $1 \times 3 + 5 \times 6 - 7 \times 3$,

„ (iii), $1 \times 3 - 7 \times 3$,

„ (iv), 5×1,

and therefore

$$A = (3 - 18 + 27) + (3 + 30 - 21) + (3 - 21) + 5,$$
$$= 11.$$

The compositions of 50 are three in number, viz.

$$1^2 + 7^2,$$
$$7^2 + 1^2,$$
$$5^2 + 5^2,$$

and the rule gives

$$B = \quad 1 \times -7,$$
$$+ \; -7 \times \quad 1,$$
$$+ \quad 5 \times \quad 5 = 11.$$

The theorem may be proved analytically by means of the formulæ

$$\text{(i)} \quad \frac{\Pi_1^\infty (1 - x^{2n})}{\Pi_1^\infty (1 - x^{2n-1})} = \Sigma_0^\infty x^{\frac{1}{2}n(n+1)},$$

$$\text{(ii)} \quad \Pi_1^\infty (1 - x^n)^3 = \Sigma_0^\infty (-)^n (2n+1)\, x^{\frac{1}{2}n(n+1)},$$

for, from (i),

$$\{\Sigma_0^\infty x^{\frac{1}{2}n(n+1)}\}^3 \times \Pi_1^\infty (1 - x^n)^3 = \Pi_1^\infty (1 - x^{2n})^6,$$

whence, by (ii),

$$\{\Sigma_0^\infty x^{\frac{1}{2}n(n+1)}\}^3 \times \Sigma_0^\infty (-)^n (2n+1)\, x^{\frac{1}{2}n(n+1)} = \{\Sigma_0^\infty (-)^n (2n+1) x^{n(n+1)}\}^2.$$

Writing x^8 for x, and multiplying throughout by x^4, this equation becomes

$$\{\Sigma_0^\infty x^{(2n+1)^2}\}^3 \times \Sigma_0^\infty (-)^n (2n+1)\, x^{(2n+1)^2} = \{\Sigma_0^\infty (-)^n (2n+1)\, x^{2(2n+1)^2}\}^2,$$

and the theorem is deducible at once from this formula by equating the coefficients of like powers of x.

THEOREMS RELATING TO THE SUM OF THE UNEVEN DIVISORS OF A NUMBER.

By J. W. L. Glaisher.

§ 1. THROUGHOUT this paper n is used to denote a number generally, *i.e.* n may denote according to circumstances an even number, an uneven number or any number; but m is always used to denote an uneven number.

The expression $\Delta(n)$ denotes the sum of the uneven divisors of n, so that if n is uneven, $\Delta(n) =$ the sum of all the divisors of n, a quantity usually denoted by $\psi(n)$. Thus, in general, if $n = 2^r m$, we have

$$\Delta(n) = \psi(m).$$

If α be is divisor of n, then the conjugate of α is α' where $\alpha\alpha' = n$, so that the product of a divisor and its conjugate is by definition always equal to the number.

The following three sections contain theorems involving the quantity $\Delta(n)$.

§ 2. *Theorem I.*

$$\Delta(1)\Delta(2n-1)+\Delta(3)\Delta(2n-1)+\Delta(5)\Delta(2n-5)\ldots+\Delta(2n-1)\Delta(1)$$

= sum of the cubes of the divisors of n that have uneven conjugates.

If $2n = 2^r m$, m being, as mentioned in § 1, uneven, and if $1, \alpha, \beta, \ldots m$ are the divisors of m, then $2^{r-1}, 2^{r-1}\alpha, 2^{r-1}\beta, \ldots 2^{r-1}m$ are the divisors of n that have uneven conjugates, and we may therefore state the theorem: if n is even and $= 2^r m$, then

$$\Delta(1)\Delta(n-1)+\Delta(3)\Delta(n-3)+\Delta(5)\Delta(n-5)\ldots+\Delta(n-1)\Delta(1)$$

$= 2^{3r-3} \times$ sum of the cubes of the uneven divisors of n.

If $r = 1$, so that n is the double of an uneven number $= 2m$, then the expression on the left-hand side contains a middle term, and we may write the formula

$$2\{\Delta(1)\Delta(n-1)+\Delta(3)\Delta(n-3)\ldots+\Delta(\tfrac{1}{2}n-2)\Delta(\tfrac{1}{2}n+2)\}+\Delta(\tfrac{1}{2}n)\Delta(\tfrac{1}{2}n)$$

$= 2^{3r-3} \times$ sum of the cubes of the uneven divisors of n;

but if n is divisible by 4, so that $n = 2^r m$ where $r > 1$, then there are two middle terms, and the formula may be written

$$\Delta(1)\,\Delta(n-1) + \Delta(3)\,\Delta(n-3) \ldots + \Delta(\tfrac{1}{2}n-1)\,\Delta(\tfrac{1}{2}n+1)$$

$= 2^{3r-4} \times$ sum of cubes of uneven divisors of n.

Taking as examples the cases $n = 10$ and $n = 12$, since

$$\Delta(1) = 1, \quad \Delta(5) = 6, \quad \Delta(9) = 13,$$
$$\Delta(3) = 4, \quad \Delta(7) = 8, \quad \Delta(11) = 12,$$

we find from the former of the two last-written formulæ, for $n = 10 = 2.5$,

$$2\{1.13 + 4.8\} + 6.6 = 1^3 + 5^3,$$

that is
$$90 + 36 = 1 + 125,$$

and from the latter, for $n = 12 = 2^2.3$,

$$1.12 + 4.13 + 6.8 = 2^2 \times (1^3 + 3^3),$$

that is
$$12 + 52 + 48 = 4 \times 28.$$

§ 3. *Theorem II.*

$$\Delta(2n+1) + 24\Delta(1)\,\Delta(2n) + 24\Delta(2)\,\Delta(2n-1) + \ldots$$
$$+ 24\Delta(2n)\,\Delta(1) + \Delta(2n+1)$$

= twice the sum of the cubes of the divisors of $2n+1$.

There are always two middle terms, and we may therefore write the theorem in the form: if n is uneven, then

$$\Delta(n) + 24\Delta(1)\,\Delta(n-1) + 24\Delta(2)\,\Delta(n-2) + \ldots$$
$$+ 24\Delta\{\tfrac{1}{2}(n-1)\}\,\Delta\{\tfrac{1}{2}(n+1)\}$$

= sum of the cubes of the divisors of n.

As an example, taking $n = 9$, since

$$\Delta(1) = 1, \quad \Delta(4) = 1, \quad \Delta(7) = 8,$$
$$\Delta(2) = 1, \quad \Delta(5) = 6, \quad \Delta(8) = 1,$$
$$\Delta(3) = 4, \quad \Delta(6) = 4, \quad \Delta(9) = 13,$$

this formula gives

$$13 + 24\{1.1 + 1.8 + 4.4 + 1.6\} = 1^3 + 3^3 + 9^3;$$

that is,
$$13 + 744 = 1 + 27 + 729.$$

§ 4. *Theorem III.*

$$\Delta(2n) + 24\Delta(2)\,\Delta(2n-2) + 24\Delta(4)\,\Delta(2n-4) + \ldots + 24\Delta(2n-2)\,\Delta(2) + \Delta(2n) = \tfrac{1}{3}(2A - 2B - C),$$

where A = sum of cubes of even divisors of $2n$,

B = „ uneven „

C = „ divisors having uneven conjugates.

If $2n = 2^r m$, and if (as in § 2) $1, \alpha, \beta, \ldots m$ are the divisors of m, then the even divisors of $2n$ are

$$\begin{array}{l} 2\ ,\ 2^2\ ,\ 2^3\ ,\ \ldots 2^r\ , \\ 2\alpha\ ,\ 2^2\alpha\ ,\ 2^3\alpha\ ,\ \ldots 2^r\alpha\ , \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 2m,\ 2^2m,\ 2^3m,\ \ldots 2^rm, \end{array}$$

the uneven divisors are

$$1,\ \alpha,\ \beta,\ \ldots m,$$

and the divisors with uneven conjugates are

$$2^r,\ 2^r\alpha,\ 2^r\beta,\ \ldots 2^r m.$$

Thus

$$\tfrac{1}{3}(2A - 2B - C)$$

$$= \tfrac{1}{3}\Sigma\left\{2\left(2^3 + 2^6 \ldots + 2^{3r}\right) - 2 - 2^{3r}\right\}\alpha^3$$

$$= \tfrac{1}{3}\Sigma\,\frac{2^{3r+3} + 2^{3r} - 2^5 + 2}{2^3 - 1}\,\alpha^3$$

$$= \frac{1}{3}\,\frac{2^{3r} \times 9 - 30}{7}\,\Sigma\alpha^3 = \frac{2^{3r} \times 3 - 10}{7}\,\Sigma\alpha^3.$$

Thus, if n is even,

$$\Delta(n) + 24\Delta(2)\,\Delta(n-2) + 24\Delta(4)\,\Delta(n-4) + \ldots + 24\Delta(n-2)\,\Delta(2) + \Delta(n)$$

$= \frac{1}{7}\{2^{3r} \times 3 - 10\} \times$ sum of the cubes of the uneven divisors of n.

If n is the double of an uneven number $= 2m$, there are two middle terms, and the formula may be written

$$\Delta(n) + 24\Delta(2)\,\Delta(n-2) \ldots + 24\Delta\left(\tfrac{1}{2}n - 1\right)\Delta\left(\tfrac{1}{2}n + 1\right)$$

$= \frac{1}{7}\{2^{3r-1} \times 3 - 5\} \times$ sum of the cubes of the uneven divisors of n.

If n is divisible by 4, and $= 2^r m$ where $r > 1$, the formula becomes

$$2\{\Delta(n)+24\Delta(2)\Delta(n-2)\ldots+24\Delta(\tfrac{1}{2}n-2)\Delta(\tfrac{1}{2}n+2)\}+24\Delta(\tfrac{1}{2}n)\Delta(\tfrac{1}{2}n)$$
$$= \tfrac{1}{7}\{2^{3r}\times 3-10\} \times \text{sum of the cubes of the uneven divisors of } n.$$

Taking, as examples, $n = 10$ and $n = 12$, the two formulæ give respectively

$$6 + 24.1.1 + 24.1.4 = \tfrac{1}{7}\{2^3 \times 3 - 5\} \times (1^3 + 5^3)$$

and

$$2\{4 + 24.1.6 + 24.1.1\} + 24.4.4 = \tfrac{1}{7}\{2^6 \times 3 - 10\} \times (1^3 + 3^3).$$

§ 5. The preceding theorems were deduced from the q-series for

$$\left(\frac{2K}{\pi}\right)^2, \quad \left(\frac{2kK}{\pi}\right)^2, \quad \left(\frac{2K}{\pi}\right)^4, \quad \left(\frac{2kK}{\pi}\right)^4,$$

but some transformation is necessary in order to reduce them to the forms given. The q-series in question are most conveniently written in the forms in which they are expressed in Professor H. J. S. Smith's report on the Theory of Numbers,* viz.

$$\left(\frac{2K}{\pi}\right)^2 = 1 + 8\{2+(-1)^n\}\,\Sigma_n\Sigma_\delta\delta q^n, \quad \left(\frac{2kK}{\pi}\right)^2 = 16\Sigma_\nu\Sigma_\delta\delta q^\nu,$$

$$\left(\frac{2K}{\pi}\right)^4 = 1 + 16\Sigma_n\Sigma_d(-)^{n+d}d^3q^n, \qquad \left(\frac{2kK}{\pi}\right)^4 = 256\Sigma_n\Sigma_{\delta'}\delta'^3q^{2n},$$

where n denotes any number, ν any uneven number, d any divisor of n, δ any uneven divisor of n, and δ' is conjugate to δ.

§ 6. The first theorem would be useful in verifying the accuracy of a factor table of $\psi(n)$, the sum of the divisors of n for uneven values of n; for, n being even, the theorem may be written

$$\psi(1)\psi(n-1) + \psi(3)\psi(n-3)\ldots+\psi(n-1)\psi(1)$$
$$= \text{sum of the cubes of the divisors of } n \text{ that have uneven conjugates.}$$

In his paper *Observatio de summis divisorum*† Euler gave a table of $\psi(n)$ as far as $n = 100$; and taking the values from

* *British Association Report* for 1865, p. 336.
† *Opera Minora Collecta*, vol. I. p. 308.

this table, we find

$(1, 99) = 1.156 = 156$,	$(25, 75) = 31.124 = 3844$,
$(3, 97) = 4.98 = 392$,	$(27, 73) = 40.74 = 2960$,
$(5, 95) = 6.120 = 720$,	$(29, 71) = 30.72 = 2160$,
$(7, 93) = 8.128 = 1024$,	$(31, 69) = 32.96 = 3072$,
$(9, 91) = 13.112 = 1456$,	$(33, 67) = 48.68 = 3264$,
$(11, 89) = 12.90 = 1080$,	$(35, 65) = 48.84 = 4032$,
$(13, 87) = 14.120 = 1680$,	$(37, 63) = 38.104 = 3952$,
$(15, 85) = 24.108 = 2592$,	$(39, 61) = 56.62 = 3472$,
$(17, 83) = 18.84 = 1512$,	$(41, 59) = 42.60 = 2520$,
$(19, 81) = 20.121 = 2420$,	$(43, 57) = 44.80 = 3520$,
$(21, 79) = 32.80 = 2560$,	$(45, 55) = 78.72 = 5616$,
$(23, 77) = 24.96 = 2304$,	$(47, 53) = 48.52 = 2592$,
	$(49, 51) = 57.72 = 4104$,

where, for brevity, (m, n) is written in place of $\psi(m)\,\psi(n)$. The sum of these terms $= 63{,}004$, and this number, multiplied by 2, $= 126{,}008$, which is the value of the expression

$$\psi(1)\,\psi(99) + \psi(3)\,\psi(97) \ldots + \psi(99)\,\psi(1).$$

Now $100 = 2^2.25$, and the uneven divisors of 100 are 1, 5, 25, so that

$2^{3r-3} \times$ sum of the cubes of the uneven divisors of n

$$= 2^3 \times (1^3 + 5^3 + 25^3)$$

$$= 8 \times (1 + 125 + 15625) = 126{,}008.$$

I was led to the theorems in §§2–4 in seeking for formulæ of verification for an extended table of $\psi(n)$ which I have constructed. Two other formulæ are available for such a purpose, viz. Euler's formula

$$\psi(n) - \psi(n-1) - \psi(n-2) + \psi(n-5) + \psi(n-7) - \ldots = 0,$$

where 1, 2, 5, 7, ... are the pentagonal numbers $\frac{1}{2}n(3n \pm 1)$ and $\psi(n-n)$ is to be replaced by n; and the formula

$$\psi(n) - 3\psi(n-1) + 5\psi(n-3) - 7\psi(n-6) + 9\psi(n-10) - \ldots = 0,$$

where 1, 3, 6, 10, ... are the triangular numbers, and $\psi(n-n)$

is to be replaced by $\frac{1}{3}n$. In both cases the formulæ end at the term preceding the first term with a negative argument.

For example, taking $n = 10$, the second formula gives

$$18 - 3.13 + 5.8 - 7.7 + 9.\tfrac{1}{3}10 = 0,$$

and for $n = 11$,

$$12 - 3.18 + 5.15 - 7.6 + 9 = 0.$$

These formulæ verify the table for even as well as uneven arguments, but the verification afforded is not so systematic.

§ 7. I have obtained another formula of this kind, involving sums of divisors of uneven numbers, by the following process.

From the formula

$$\sqrt{\left(\frac{2K}{\pi}\right)} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + 2q^{\frac{49}{4}} + \&c.$$

we deduce

$$\left(\frac{2K}{\pi}\right)^2 = 16q\,(1 + q^2 + q^6 + q^{12} + q^{20} + \&c.)^4,$$

the exponents being the doubles of the triangular numbers, *i.e.* the numbers of the form $n\,(n+1)$, whence we obtain the identical equation

$$\psi(1)\,q + \psi(3)\,q^3 + \psi(5)\,q^5 + \psi(7)\,q^7 + \&c. = q\,(1 + q^2 + q^6 + q^{12} + q^{20} + \&c.)^4.$$

Differentiating logarithmically,

$$\frac{\psi(1)\,q + 3\psi(3)\,q^3 + 5\psi(5)\,q^5 + \&c.}{\psi(1)\,q + \psi(3)\,q^3 + \psi(5)\,q^5 + \&c.} = 1 + \frac{4(2q^2 + 6q^6 + 12q^{12} + \&c.)}{1 + q^2 + q^6 + q^{12} + \&c.}$$

$$= \frac{1 + 9q^2 + 25q^6 + 49q^{12} + 81q^{20} + \&c.}{1 + q^2 + q^6 + q^{12} + q^{20} + \&c.},$$

whence

$$\Sigma_1^\infty q^{n(n-1)} \times \Sigma_1^\infty (2n-1)\,\psi(2n-1)\,q^{2n-1} = \Sigma_1^\infty (2n-1)\,q^{n(n-1)} \times \Sigma_1^\infty \psi(2n-1)\,q^{2n-1}.$$

Equating the coefficients of q^{2n+1},

$$(2n+1)\,\psi(2n+1) + (2n-1)\,\psi(2n-1) + (2n-5)\,\psi(2n-5) + (2n-11)\,\psi(2n-11) + \&c.$$

$$= \psi(2n+1) + 9\psi(2n-1) + 25\psi(2n-5) + 49\psi(2n-11) + \&c.,$$

whence

$$2n\psi(2n+1)+(2n-10)\psi(2n-1)+(2n-30)\psi(2n-5)$$
$$+(2n-60)\psi(2n-11)+\&c.=0,$$

and therefore, dividing by 2,

$$n\psi(2n+1)+(n-5)\psi(2n-1)+(n-15)\psi(2n-5)$$
$$+(n-30)\psi(2n-11)+\&c.=0,$$

which is the resulting formula. The differences between the arguments in the successive terms are the even numbers 2, 4, 6, 8, &c., and the differences between the successive multipliers are the multiples of five, 5, 10, 15, &c. The series ends at the term preceding the first term with zero or negative argument, and $\psi(0)=0$.

As an example, putting $2n+1=49$, the formula gives

$$24\psi(49)+19\psi(47)+9\psi(43)-6\psi(37)-26\psi(29)$$
$$-51\psi(19)-81\psi(7)=0,$$

that is,

$$24\times 57+19\times 48+9\times 44-6\times 38-26\times 30-51\times 20-81\times 8=0,$$

which is true, the sums of the positive and negative terms being each equal to 2676.

The theorem may be written also in the form

$$n\{\psi(2n+1)+\psi(2n-1)+\psi(2n-5)+\psi(2n-11)+\&c.\}$$
$$=5\{\psi(2n+1)+3\psi(2n-1)+6\psi(2n-5)+10\psi(2n-11)+\&c.\}.$$

§ 8. Similarly by means of the formula for $\left(\frac{2K}{\pi}\right)^2$ we obtain the equation

$$1+8\Sigma_1^\infty\{2+(-1)^n\}\Delta(n)q^n=(1+2q+2q^4+2q^9+\&c.)^4,$$

whence, following the same process as in the last section, we find

$$\{1+2\Sigma_1^\infty q^{n^2}\}\times\Sigma\{2+(-1)^n\}n\Delta(n)q^n$$
$$=\Sigma_1^\infty n^2q^{n^2}\times[1+8\Sigma_1^\infty\{2+(-1)^n\}\Delta(n)q^n],$$

and equating the coefficients of q^{n+1}, we obtain finally

$$\varepsilon_n n\Delta(n)+\varepsilon_{n-1}(2n-10)\Delta(n-1)+\varepsilon_{n-4}(2n-40)\Delta(n-4)$$
$$+\varepsilon_{n-9}(2n-90)\Delta(n-9)+\varepsilon_{n-16}(2n-160)\Delta(n-16)$$
$$+\varepsilon_{n-25}(2n-250)\Delta(n-25)+\&c.=0,$$

where ε_m denotes 3 or 1 according as m is even or uneven. The series terminates when the arguments become negative or zero, and $\Delta(0) = 0$.

For example, for $n = 20$ the formula gives

$$3 \times 20\Delta(20) + 30\Delta(19) + 0 - 50\Delta(11) - 3 \times 120\Delta(4),$$

that is,

$$60.6 + 30.20 - 50.12 - 360.1 = 0,$$

and, for $n = 19$,

$$19\Delta(19) + 3 \times 28\Delta(18) - 2\Delta(15) - 3 \times 52\Delta(10) - 122 \times \Delta(3) = 0,$$

that is,

$$19.20 + 84.13 - 2.24 - 156.6 - 122.4 = 0.$$

ON MR. ANGLIN'S FORMULA FOR THE SUCCESSIVE POWERS OF THE ROOT OF AN ALGEBRAICAL EQUATION.

By Professor Cayley.

SUPPOSE $x^m - px^{m-1} + qx^{m-2} - \ldots = 0$, then the successive powers x^m, x^{m+1}, x^{m+2}, &c. of x can be expressed in the form $Px^{m-1} - Qx^{m-2} + Rx^{m-3} - \&c.$ Mr. Anglin has obtained for this purpose a very elegant formula, with a demonstration, which (it occurred to me) might be presented under a somewhat simplified form; and he has permitted me to draw up the present Note.

Take, for greater convenience, the equation to be

$$x^4 - px^3 + qx^2 - rx + s = 0,$$

and let h_1, h_2, h_3, ... be the sums of the homogeneous products of the roots, of the orders 1, 2, 3, &c. respectively; then, writing also $h_0 = 1$, we have

$$\begin{aligned}
h_1 &= h_0 p, \\
h_2 &= h_1 p - h_0 q, \\
h_3 &= h_2 p - h_1 q + h_0 r, \\
h_4 &= h_3 p - h_2 q + h_1 r - h_0 s, \\
h_5 &= h_4 p - h_3 q + h_2 r - h_1 s, \\
&\vdots
\end{aligned}$$

And this being so, starting from the equation

$$x^4 = px^3 - qx^2 + rx - s,$$

that is,

$$= h_1x^3 - h_0qx^2 + h_0rx - h_0s,$$

we obtain successively

$$\begin{aligned}
x^5 = \;& h_1(px^3 - qx^2 + rx - s) \\
& - h_0qx^3 + h_0rx^2 - h_0sx \\
= \;& h_2x^3 - (h_1q - h_0r)\,x^2 + (h_1r - h_0s)\,x - h_1s, \\
x^6 = \;& h_2(px^3 - qx^2 + rx - s) \\
& - (h_1q - h_0r)\,x^3 + (h_1r - h_0s)\,x^2 - h_1sx \\
= \;& h_3x^3 - (h_2q - h_1r + h_0s)\,x^2 + (h_2r - h_1s)\,x - h_2s, \\
x^7 = \;& h_3(px^3 - qx^2 + rx - s) \\
& - (h_2q - h_1r + h_0s)\,x^3 + (h_2r - h_1s)\,x^2 - h_2sx \\
= \;& h\,x^3 - (h_3q - h_2r + h_1s)\,x^2 + (h_3r - h_2s)\,x - h_3s,
\end{aligned}$$

and so on, the characteristic feature being that by the introduction of the symbols h, the coefficient of x^3 presents itself at each step as a monomial, and the coefficients of the lower powers require no reduction. It is obvious that the process is a perfectly general one, and that for the equation

$$x^m - p_1x^{m-1} + p_2x^{m-2} \ldots + (-)^m p_m = 0,$$

the formula is

$$\begin{aligned}
x^{m+\theta} = \;& h_{\theta+1}x^{m-1} \\
& - (h_\theta p_2 - h_{\theta-1}p_3 + \ldots)\,x^{m-2} \\
& + (h_\theta p_3 - h_{\theta-1}p_4 + \ldots)\,x^{m-3} \\
& \quad\vdots \\
& + (-)^{s-1}(h_\theta p_s - h_{\theta-1}p_{s+1} + \ldots)\,x^{m-s} \\
& \quad\vdots \\
& + (-)^{m-1}.h_\theta p_m \qquad\qquad x^0,
\end{aligned}$$

where, as regards each power of x, the series forming the coefficient thereof is continued as far as possible, that is, up to the term which contains p_m or h_0 as the case may be.

A REVISION OF CHAPTERS XXIV. AND XXVI. OF LEGENDRE'S FONCTIONS ELLIPTIQUES t. I.

By A. G. Greenhill, M.A.

THE great simplification introduced into the theory and notation of elliptic integrals of the third kind by the introduction of Jacobi's notation for the elliptic functions has been pointed out by Prof. Cayley in Chapter V. of his *Elliptic Functions*.

In reading Legendre's *Fonctions Elliptiques* it is useful to keep Jacobi's notation in mind, and the following notes on Chapter XXIV. and XXVI. are intended to illustrate the kind of revision required, keeping at the same time to Legendre's notation in the text.

Chapter XXIV.

118. If $$p = \frac{\sin\phi\cos\phi}{(1+\zeta\sin^2\phi)\,\Delta\phi},$$

and if $$\sin\phi = \operatorname{sn} u,$$

$$\Delta\phi = \sqrt{(1-c^2\sin^2\phi)} = \operatorname{dn} u;$$

then, if k denote an arbitrary constant,

$$\frac{dp}{1+kp^2} = \frac{d\phi}{\Delta\phi}\,\frac{N}{(1+m\sin^2\phi)\,(1+m'\sin^2\phi)\,(1+n\sin^2\phi)},$$

if
$$\left.\begin{aligned} nn'm &= -c^2\zeta^2 \\ (1+n)\,(1+n')\,(1+m) &= b^2(1+\zeta)^2 \\ (c^2+n)\,(c^2+n')\,(c^2+m) &= -b^2c^2k \end{aligned}\right\} \ldots\ldots(1),$$

where $$b^2 + c^2 = 1.$$

Now put $$n = -c^2\operatorname{sn}^2 a,$$

$$n' = -c^2\operatorname{sn}^2 a',$$

$$m = -c^2\operatorname{sn}^2 b;$$

then
$$1+n = \mathrm{dn}^2 a,$$
$$1+n' = \mathrm{dn}^2 a',$$
$$1+m = \mathrm{dn}^2 b;$$

and
$$c^2+n = c^2\,\mathrm{cn}^2 a,$$
$$c^2+n' = c^2\,\mathrm{cn}^2 a',$$
$$c^2+m = c^2\,\mathrm{cn}^2 b.$$

Therefore equations (1) become

$$\left.\begin{aligned} c^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 a'\,\mathrm{sn}^2 b &= \zeta^2 \\ \mathrm{dn}^2 a\,\mathrm{dn}^2 a'\,\mathrm{dn}^2 b &= b^2(1+\zeta)^2 \\ c^2\,\mathrm{cn}^2 a\,\mathrm{cn}^2 a'\,\mathrm{cn}^2 b &= -b^2 k \end{aligned}\right\} \ldots\ldots\ldots(1)^*.$$

The elimination of m between the first two of equations (1) gives

$$1 = \frac{c^2\zeta^2}{nn'} + \frac{b^2(1+\zeta)^2}{(1+n)(1+n')},$$

whence

$$\zeta = \frac{-b^2 \pm \sqrt{\left\{\left(\frac{1+n}{n}\right)\left(\frac{1+n'}{n'}\right)(c^2+n)(c^2+n')\right\}}}{b^2 + c^2 \cdot \frac{1+n}{n} \cdot \frac{1+n'}{n'}} \ldots(2),$$

and therefore

$$\zeta = \frac{-b^2 \pm \frac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{\mathrm{sn}\,a} \cdot \frac{\mathrm{cn}\,a'\,\mathrm{dn}\,a'}{\mathrm{sn}\,a'}}{b^2 + \frac{1}{c^2} \cdot \frac{\mathrm{dn}^2 a}{\mathrm{sn}^2 a} \cdot \frac{\mathrm{dn}^2 a'}{\mathrm{sn}^2 a'}} \ldots\ldots\ldots\ldots(2)^*$$

$$= c^2\,\mathrm{sn}\,a\,\mathrm{sn}\,a' \frac{\mathrm{cn}\,(a+a')}{\mathrm{dn}\,(a+a')},$$

or
$$= -c^2\,\mathrm{sn}\,a\,\mathrm{sn}\,a' \frac{\mathrm{cn}\,(a-a')}{\mathrm{dn}\,(a-a')};$$

the second value being derived from the first by changing a' into $-a'$.

Then, finally, taking the first value of ζ,

$$m = -\frac{c^2\zeta^2}{nn'} = -\frac{\zeta^2}{c^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 a'}$$

$$= -c^2\frac{\mathrm{cn}^2(a+a')}{\mathrm{dn}^2(a+a')},$$

$$k = -(c^2+n)(c^2+n')\frac{c^2+m}{b^2c^2}$$

$$= -c^4\,\mathrm{cn}^2 a\,\mathrm{cn}^2 a'\frac{\mathrm{sn}^2(a+a')}{\mathrm{dn}^2(a+a')}.$$

Then $$\mathrm{sn}^2 b = \frac{\mathrm{cn}^2(a+a')}{\mathrm{dn}^2(a+a')},$$

giving $$a+a'-b = K,$$

and $$\zeta = -c^2\,\mathrm{sn}\,a\,\mathrm{sn}\,a'\,\mathrm{sn}\,b.$$

Since $$\frac{d\phi}{\Delta\phi} = du,$$

therefore

$$\frac{dp}{1+kp^2} = \frac{du}{\zeta} + \frac{A\,du}{1+n\,\mathrm{sn}^2 u} + \frac{A'du}{1+n'\,\mathrm{sn}^2 u} + \frac{B\,du}{1+m\,\mathrm{sn}^2 u} \ldots (3),$$

where, with the above values,

$$\left.\begin{aligned} A &= \frac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{\sqrt{(-k)}\,\mathrm{sn}\,a} \\ A' &= \frac{\mathrm{cn}\,a'\,\mathrm{dn}\,a'}{\sqrt{(-k)}\,\mathrm{sn}\,a'} \\ B &= -\frac{\mathrm{cn}\,b\,\mathrm{dn}\,b}{\sqrt{(-k)}\,\mathrm{sn}\,b} \end{aligned}\right\} \ldots\ldots\ldots\ldots (3)^*,$$

and $$\sqrt{(-k)} = c^2\,\mathrm{cn}\,a\,\mathrm{cn}\,a'\frac{\mathrm{sn}(a+a')}{\mathrm{dn}(a+a')}$$

$$= -\frac{c^2}{c'}\,\mathrm{cn}\,a\,\mathrm{cn}\,a'\,\mathrm{cn}\,b,$$

if $$c^2 + c'^2 = 1.$$

With Legendre's notation,

$$\Pi n = \int_0 \frac{1}{1+n\sin^2\phi} \cdot \frac{d\phi}{\Delta\phi}$$

$$= \int_0 \frac{du}{1+n\,\mathrm{sn}^2 u};$$

but, in Jacobi's notation,

$$\Pi(u, a) = \int_0 \frac{c^2\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2 u}{1-c^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u}\,du$$

$$= uZa + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

so that

$$\sqrt{(-k)}\,A\Pi n = \int_0 \frac{\frac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{\mathrm{sn}\,a}}{1-c^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u}\,du = u\frac{\mathrm{cn}\,a\,\mathrm{dn}\,u}{\mathrm{sn}\,a} + \Pi(u, a);$$

and Legendre's equation

$$A\Pi n + A'\Pi n' + B\Pi m + \frac{1}{\zeta}F = \int\frac{dp}{1+kp^2} \quad \ldots\ldots\ldots (4)$$

becomes, with Jacobi's notation,

$$\Pi(u, a) + \Pi(u, a') - \Pi(u, b) + u\sqrt{(-k)}$$

$$= \int\frac{\sqrt{(-k)}\,dp}{1+kp^2}$$

$$= \tfrac{1}{2}\log\frac{1+p\sqrt{(-k)}}{1-p\sqrt{(-k)}} \quad \ldots (4)^*,$$

where $$b = a + a' - K,$$

the formula for the addition of parameters.

119. Putting $m=-1$, is equivalent to putting $\zeta=-1$ and $b=iK'$, as in formula (f'); and so also $m=0$, gives $\zeta=0$ and $b=0$, formula (g'); the particular cases discussed in Chapter XV.

If the parameters n and n' are each of the form $-1+b^2\sin^2\mu$, then a and a' are each of the form $ia+K$; and so also is b.

If n and n' are each of the form $-c^2\sin^2\lambda$, then a and a' are real, and so also is b.

But if n is of the form $-1+b^2\sin^2\mu$ and n' of the form $-c^2\sin^2\lambda$, then a is of the form $ia+K$, but a' is real, so that b and therefore m is imaginary.

Formula (f')

$$\Pi n + \Pi\frac{c^2}{n} = F + \frac{1}{\sqrt{\alpha}}\arctan\frac{\sqrt{\alpha}\tan\phi}{\Delta}$$

$$= F + \frac{1}{2\sqrt{(-\alpha)}}\log\frac{\Delta\phi + \sqrt{(-\alpha)}\tan\phi}{\Delta\phi - \sqrt{(-\alpha)}\tan\phi}$$

becomes, in Jacobi's notation

$$\Pi(u, a+iK') = \Pi(u, a) + u\frac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{\mathrm{sn}\,a} + \tfrac{1}{2}\log\frac{\mathrm{sn}(a-u)}{\mathrm{sn}(a+u)} \ldots (f')^*.$$

But, if $n = -c^2\,\mathrm{sn}^2 a$,

$$\Pi n = \int_0^u \frac{du}{1 + n\,\mathrm{sn}^2 u}$$

$$= \int_0^u \frac{du}{1 - c^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u},$$

then

$$\Pi\frac{c^2}{n} = \int_0^u \frac{du}{1 - \dfrac{\mathrm{sn}^2 u}{\mathrm{sn}^2 a}};$$

and, with Jacobi's notation,

$$\int_0^u \frac{\dfrac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{\mathrm{sn}\,a}}{1 - \dfrac{\mathrm{sn}^2 u}{\mathrm{sn}^2 a}} \quad (-a < u < a) \quad = \tfrac{1}{2}\log\frac{\Theta_1(a+u)}{\Theta_1(a-u)} - uZa;$$

$$\int_u^K \frac{\dfrac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{\mathrm{sn}\,a}}{\dfrac{\mathrm{sn}^2 u}{\mathrm{sn}^2 a} - 1} (a < u < 2K - a) = \tfrac{1}{2}\log\frac{\Theta_1(u+a)}{\Theta_1(u-a)} + (K-u)\,Za.$$

With this notation, equation (f') is satisfied identically.

Similarly, equation (g') interpreted in Jacobi's notation, gives

$$\Pi(u, a+K) = \Pi(u, a) - u\frac{c^2\,\mathrm{sn}\,a\,\mathrm{cn}\,a}{\mathrm{dn}\,a} + \tfrac{1}{2}\log\frac{\mathrm{dn}(a-u)}{\mathrm{dn}(a+u)} \ldots\ldots\ldots\ldots (g').^*$$

120. If the parameter n and n' are conjugate imaginaries; that is, if

$$n = \nu(\cos\theta + i\sin\theta) = \nu e^{i\theta},$$
$$n' = \nu(\cos\theta - i\sin\theta) = \nu e^{-i\theta},$$

when i is used to denote $\sqrt{(-1)}$, then the two values of m will be real.

The values of a and a' must also be conjugate imaginaries, so that if

$$a = \tfrac{1}{2}(\alpha + i\beta), \quad \text{then} \quad a' = \tfrac{1}{2}(\alpha - i\beta).$$

Then $$\nu^2 = nn' = c^4 \operatorname{sn}^2 a \operatorname{sn}^2 a';$$

$$\nu = c^2 \operatorname{sn} a \operatorname{sn} a';$$

and $$h = \frac{b^2\nu}{\nu^2 + 2c^2\nu\cos\theta + c^2}$$

$$= \frac{b^2 \operatorname{sn} a \operatorname{sn} a'}{1 - c^2\operatorname{sn}^2 a - c^2\operatorname{sn}^2 a' + c^2\operatorname{sn}^2 a \operatorname{sn}^2 a'}$$

$$= \tfrac{1}{2}\frac{\operatorname{cn}(a-a')\operatorname{dn}(a+a') - \operatorname{cn}(a+a')\operatorname{dn}(a-a')}{\operatorname{dn}(a+a')\operatorname{dn}(a-a')}$$

$$= \tfrac{1}{2}\frac{\operatorname{cn}(a-a')}{\operatorname{dn}(a-a')} - \tfrac{1}{2}\frac{\operatorname{cn}(a+a')}{\operatorname{dn}(a+a')}$$

$$= \tfrac{1}{2}\frac{\operatorname{cn} i\beta}{\operatorname{dn} i\beta} - \tfrac{1}{2}\frac{\operatorname{cn}\alpha}{\operatorname{dn}\alpha}.$$

Also $$\zeta = -\nu h \pm \nu\sqrt{(1 + 2h\cos\theta + h^2)};$$

therefore

$$\sqrt{(1 + 2h\cos\theta + h^2)} = \tfrac{1}{2}\frac{\operatorname{cn}(a-a')}{\operatorname{dn}(a-a')} + \tfrac{1}{2}\frac{\operatorname{cn}(a+a')}{\operatorname{dn}(a+a')}$$
$$= \tfrac{1}{2}\frac{\operatorname{cn} i\beta}{\operatorname{dn} i\beta} + \tfrac{1}{2}\frac{\operatorname{cn}\alpha}{\operatorname{dn}\alpha}.$$

If m and m' denote the two values of m, then

$$m = -c^2\frac{\operatorname{cn}^2(a+a')}{\operatorname{dn}^2(a+a')} = -c^2\frac{\operatorname{cn}^2\alpha}{\operatorname{dn}^2\alpha} = -c^2\operatorname{sn}^2(\alpha - K),$$

$$m' = -c^2\frac{\operatorname{cn}^2(a-a')}{\operatorname{dn}^2(a-a')} = -c^2\frac{\operatorname{cn}^2 i\beta}{\operatorname{dn}^2 i\beta} - c^2\operatorname{sn}^2(i\beta - K),$$

and
$$m + c^2 = b^2 c^2 \frac{\mathrm{sn}^2\alpha}{\mathrm{dn}^2\alpha},$$
$$m' + c^2 = b^2 c^2 \frac{\mathrm{sn}^2 i\beta}{\mathrm{dn}^2 i\beta};$$

so that $m' + c^2$ is negative.

Putting, therefore,

$$m = -c^2 \sin^2\lambda, \quad m' = -1 + b^2 \sin^2\mu,$$

then
$$\sin\lambda = \frac{\zeta}{\nu} = -h + \sqrt{(1 + 2h\cos\theta + h^2)}$$
$$= \frac{\mathrm{cn}\,\alpha}{\mathrm{dn}\,\alpha} = \mathrm{sn}\,(\alpha + K).$$

And
$$m' + c^2 = -b^2\cos^2\mu$$
$$= b^2 c^2 \frac{\mathrm{sn}^2 i\beta}{\mathrm{dn}^2 i\beta},$$

therefore
$$\cos^2\mu = -c^2 \frac{\mathrm{sn}^2 i\beta}{\mathrm{dn}^2 i\beta}$$
$$= c^2 \frac{\mathrm{sn}^2(\beta, b)}{\mathrm{dn}^2(\beta, b)}$$
$$= \mathrm{cn}^2(K' + \beta, b),$$

so that we may put

$$\cos\mu = \mathrm{cn}\,(\beta + K', b),$$
$$\sin\mu = \mathrm{sn}\,(\beta + K', b),$$

or
$$\mu = \mathrm{am}\,(\beta + K', b),$$

K' in Jacobi's notation being used to denote $F^1 b$.

Also

$$4h^2\sin^2\theta = \frac{b^2}{c^2}\cos^2\lambda\cos^2\mu$$
$$= \frac{b^2}{c^2}\mathrm{cn}^2(\alpha - K)\,\mathrm{cn}^2(\beta - K', b)$$
$$= -b^4 \frac{\mathrm{sn}^2\alpha}{\mathrm{dn}^2\alpha} \cdot \frac{\mathrm{sn}^2 i\beta}{\mathrm{dn}^2 i\beta}.$$

We have put $m = -c^2 \operatorname{sn}^2 b$,

so that the two values of b are given by

$$b = \alpha - K,$$

and

$$b = i\beta - K.$$

The letter b is here unavoidably used in two senses, first in putting $m = -c^2 \operatorname{sn}^2 b$, and secondly, as in the text of Legendre, as the modulus complimentary to c.

121. (i) Taking the first value of m, namely

$$m = -\,c^2 \sin^2\lambda,$$

then equation (5) of Legendre shows how, when translated into Jacobi's notation, to make the sum

$$\Pi\{u, \tfrac{1}{2}(\alpha + i\beta)\} + \Pi\{u, \tfrac{1}{2}(\alpha - i\beta)\},$$

by the addition of parameters, to depend upon $\Pi(u, \alpha - K)$.

(ii) Taking the second value of m, namely

$$m' = -\,1 + b^2 \sin^2\mu,$$

then equation (6) is equivalent to showing how the difference

$$\Pi\{u, \tfrac{1}{2}(\alpha + i\beta)\} - \Pi\{u, \tfrac{1}{2}(\alpha - i\beta)\}$$

depends on $\Pi(u, i\beta - K)$.

The combination of these two cases shows how to express any integral of the form (7) involving conjugate imaginary parameters

$$\nu(\cos\theta + i\sin\theta) \quad \text{and} \quad \nu(\cos\theta - i\sin\theta)$$

in Legendre's form, or $\frac{1}{2}(\alpha + i\beta)$ and $\frac{1}{2}(\alpha - i\beta)$ in Jacobi's form, in terms of $\Pi(u, \alpha)$ and $\Pi(u, i\beta)$.

Also, with this notation,

$$f = \frac{2\,\mathrm{dn}\,\alpha}{\mathrm{dn}\,i\beta - \mathrm{dn}\,\alpha},$$

with a similar value for g; and the values of f' and g' are obtained by interchanging α and $i\beta$.

123. When $\nu = c$, then

$$n = c(\cos\theta + i\sin\theta) = ce^{i\theta},$$

$$n' = c(\cos\theta - i\sin\theta) = ce^{-i\theta};$$

and

$$nn' = c^2,$$

$$n + n' = 2c\cos\theta.$$

But if

$$a = \tfrac{1}{2}(\alpha + i\beta),$$

then

$$a' = \tfrac{1}{2}(\alpha - i\beta),$$

and $$\mathrm{sn}^2\tfrac{1}{2}(\alpha+i\beta)=-\frac{1}{c}e^{i\theta},$$

$$\mathrm{sn}^2\tfrac{1}{2}(\alpha-i\beta)=-\frac{1}{c}e^{-i\theta};$$

so that $$\mathrm{sn}^2\tfrac{1}{2}(\alpha+i\beta)\,\mathrm{sn}^2\tfrac{1}{2}(\alpha-i\beta)=\frac{1}{c^2},$$

and therefore $$\beta=K'.$$

126. When

$$n=-1+b(\cos\lambda+i\sin\lambda)=-1+be^{i\lambda},$$

$$n'=-1+b(\cos\lambda-i\sin\lambda)=-1+be^{-i\lambda};$$

then $$\mathrm{dn}^2a=be^{i\lambda},$$

$$\mathrm{dn}^2a'=be^{-i\lambda},$$

and $$\mathrm{dn}^2a\,\mathrm{dn}^2a'=b^2,$$

and $$a+a'=K;$$

so that, if $$a=\tfrac{1}{2}(\alpha+i\beta),\quad a'=\tfrac{1}{2}(\alpha-i\beta),$$

then $$\alpha=K.$$

Chapter XXVI.

Of the integral $Z=\int\frac{dz}{(1+pz^3)\sqrt[3]{(1+qz^3)}}$.

Passing over the various reductions employed by Legendre in §§ 135—140, for particular relations between p and q, begin with § 141, where the integral is considered in its generality, employing the language and notation of the text.

The particular cases and their reductions can then be mentioned incidentally.

141. Returning now to the general value of P in § 135, to make the development for all values of α; in the first place, identically,

$$\int\frac{3y^2dy}{(\alpha^3+y^3)\sqrt{(y^3-1)}}=\int\frac{dy}{(\alpha+y)\sqrt{(y^3-1)}}+\int\frac{(2y-\alpha)\,dy}{(\alpha^2-\alpha y+y^2)\sqrt{(y^3-1)}}.$$

The first member is $3Q$, which can be expressed by a circular or a logarithmic function; and if $y=1+x^2$,

$$\int\frac{2-\alpha+2x^2}{x^4+(2-\alpha)x^2+1-\alpha+\alpha^2}\cdot\frac{dx}{\sqrt{(x^4+3x^2+3)}}$$

$$=\tfrac{3}{2}Q-\int\frac{dx}{(\alpha+1+x^2)\sqrt{(x^4+3x^2+3)}}.$$

Therefore the integral X, forming the first member of this equation, can be expressed by the known quantity Q, and the integral

$$T = \int \frac{dx}{(\alpha + 1 + x^2)\sqrt{(x^4 + 3x^2 + 3)}},$$

which reduces to an elliptic integral of the third kind with a real parameter.

To determine this integral T, let, as before,

$$r^4 = 3, \quad c^2 = \frac{2 - r^2}{4}, \quad x = r \tan \tfrac{1}{2}\phi;$$

then

$$T = \frac{1}{2r} \int \frac{d\phi}{\Delta\phi} \cdot \frac{1}{\alpha + 1 + r^2 \tan^2 \tfrac{1}{2}\phi};$$

but by the development of the last factor

$$\frac{\alpha + 1 - r^2}{\alpha + 1 + r^2 \tan^2 \tfrac{1}{2}\phi} = 1 - \frac{2r^2 \{\alpha + 1 + r^2 - (\alpha + 1 - r^2)\cos\phi\}}{4r^2(\alpha + 1) + (\alpha + 1 - r^2)^2 \sin^2\phi}.$$

Let therefore

$$n = \frac{(\alpha + 1 - r^2)^2}{4r^2(\alpha + 1)},$$

the required integral

$$T = \frac{1}{2r(\alpha + 1 - r^2)} \left\{ F\phi - \frac{\alpha + 1 + r^2}{2(\alpha + 1)} \Pi n \right\} + \frac{1}{4r(\alpha + 1)} \int \frac{\cos\phi}{1 + n \sin^2\phi} \cdot \frac{d\phi}{\Delta\phi}.$$

The integral T is therefore expressible by means of the elliptic integral of the third kind Πn, of which the parameter n is real.

To introduce Jacobi's form, we must put

$$n = -\frac{1}{\mathrm{sn}^2 a},$$

$$\mathrm{sn}^2 a = -\frac{4r^2(\alpha + 1)}{(\alpha + 1 - r^2)^2},$$

$$\mathrm{cn}^2 a = \left(\frac{\alpha + 1 + r^2}{\alpha + 1 - r^2}\right)^2,$$

$$\mathrm{cn}\, a = \frac{r^2 + \alpha + 1}{r^2 + \alpha - 1},$$

But, immediately

$$T = \frac{1}{2r} \int \frac{du}{\alpha + 1 + r^2 \tan^2 \tfrac{1}{2}\phi} = \int \frac{1 + \mathrm{cn}\, u}{(\alpha + 1 - r^2)(\mathrm{cn}\, u - \mathrm{cn}\, a)}\, du,$$

which depends on the integral

$$\int_0^u \frac{du}{\operatorname{cn} u - \operatorname{cn} a},$$

and

$$\int_0^u \frac{\operatorname{sn} a \operatorname{dn} a \, du}{\operatorname{cn} u - \operatorname{cn} a}$$

$$= \tfrac{1}{2} \log \frac{1 - \operatorname{cn}(a+u)}{1 - \operatorname{cn}(a-u)} \cdot \frac{\Theta(a+u)}{\Theta(a-u)} - uZa.$$

Therefore, if $\alpha = r^2 - 1$, $\operatorname{cn} a = \infty$, and $a = iK'$; so that the elliptic integrals of the third kind can be made to disappear; or, immediately,

$$T = \frac{1}{2r^3} \int \frac{d\phi}{\Delta\phi} \cos^2 \tfrac{1}{2}\phi$$

$$= \frac{1}{4r^3} \left\{ F\phi + \int \frac{\cos\phi \, d\phi}{\sqrt{(1 - c^2 \sin^2\phi)}} \right\}.$$

When $\alpha = -1$, $\operatorname{cn} a = 1$, $a = 0$, leading to a simple result; but this would require $q = 0$.

The case of $\alpha = 0$ gives $T = \frac{1}{2} Q$, and $P = Q$, and leaves Z indeterminate, but then directly

$$Z = \frac{3}{2\sqrt{q}} \int \frac{dy}{y^2 \sqrt{(y^3 - 1)}},$$

and

$$\int \frac{dy}{y^2 \sqrt{(y^3 - 1)}} = \frac{1}{y} \sqrt{(y^3 - 1)} - \tfrac{1}{2} \int \frac{y \, dy}{\sqrt{(y^3 - 1)}},$$

and the last integral depends only on elliptic integrals of the first and second kind.

But if $\alpha = 0$,

$$\operatorname{cn} a = \frac{r^2 + 1}{r^2 - 1} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1},$$

and

$$a = \tfrac{2}{3} iK',$$

so that by Jacobi's method the integral is capable of reduction.

If $\alpha = 2$, as is §§ 136, 139; then

$$\operatorname{cn} a = \frac{r^2 + 3}{r^2 - 3} = -\frac{\sqrt{3} + 1}{\sqrt{3} - 1},$$

and

$$a = \tfrac{4}{3} iK'.$$

If $\alpha^3 = -4$, as in § 140, then

$$\operatorname{cn} a = \frac{\sqrt{3} - \sqrt[3]{4} + 1}{\sqrt{3} + \sqrt[3]{4} - 1},$$

and $$a = \tfrac{2}{3}K;$$

so that the reduction of the integral in these cases is made apparent by means of Jacobi's form.

The value of T being known, the value of X is also known, since

$$X = \tfrac{3}{2}Q - T;$$

this follows also from the method of § 121.

But this does not give the solution of the first integral Z, or of the associated integral P; the integral P must therefore be evaluated at length.

142. The substitution of $x = r\tan\frac{1}{2}\phi$ in the value of P will give, putting

$$\beta^2 = 1 - \alpha + \alpha^2,$$

$$\frac{2dx}{\sqrt{(x^4 + 3x^2 + 3)}} = \frac{1}{r}\frac{d\phi}{\Delta\phi},$$

$$\frac{1 + x^2}{x^4 + (2-\alpha)x^2 + \beta^2} = \frac{(1 + \cos\phi)^2 + r^2\sin^2\phi}{2(\beta^2+3) + \{(2-\alpha)r^2 - \beta^2 - 3\}\sin^2\phi + 2\cos\phi(\beta^2 - 3)}.$$

Numerator and denominator must now be multiplied by the quantity obtained by changing the sign of $\cos\phi$ in the denominator, and the denominator will then be

$$48\beta^2 + \{4r^2(2-\alpha)(\beta^2+3) - 48\beta^2\}\sin^2\phi + \{\beta^2 + 3 - r^2(2-\alpha)^2\}\sin^4\phi.$$

Let this denominator

$$= 48\beta^2(1 + 2\nu\cos\theta\sin^2\phi + \nu^2\sin^4\phi);$$

then

$$\nu = \frac{\beta^2 + 3 - r^2(2-\alpha)}{4\beta\nu^2} = \frac{\alpha^2 + (r^2-1)\alpha + (r^2-1)^2}{4\beta\nu^2},$$

$$\nu\cos\theta = \frac{(2-\alpha)(\beta^2+3) - 4\beta^2 r^2}{8\beta^2\nu^2},$$

$$\nu\sin\theta = \frac{\alpha(1+\alpha)(2-\alpha)}{8\beta^2}.$$

The parameters will therefore not be imaginary in three cases, namely when $\alpha = 0$, $\alpha = -1$, $\alpha = 2$; cases which have already been considered.

Let the new denominator

$$1 + 2\nu \cos\theta \sin^2\phi + \nu^2 \sin^4\phi = D;$$

from the known form of the numerator we can suppose

$$\frac{1+x^2}{x^4+(2-\alpha)x^2+\beta^2} = L + \frac{M+N\sin^2\phi}{D} + \frac{(M'+N'\sin^2\phi)\cos\phi}{D};$$

so that, by substitution and integration,

$$P = \frac{L}{r}F\phi + \int \frac{M+N\sin^2\phi}{rD}\cdot\frac{d\phi}{\Delta\phi} + \int \frac{(M'+N'\sin^2\phi)\cos\phi\, d\phi}{D\Delta\phi}.$$

The last integral depends on rational fractions, so that all depends on the integral

$$V' = \int \frac{M+N\sin^2\phi}{1+2\nu\cos\theta\sin^2\phi+\nu^2\sin^4\phi}\cdot\frac{d\phi}{\Delta\phi}.$$

It has been proved in this *Journal* (vol. XVIII. p. 66) that if the denominator D be split into factors, so that

$$\begin{aligned} D &= 1 + 2\nu\cos\theta\sin^2\phi + \nu^2\sin^4\phi \\ &= \nu^2(\sin^2\delta - \sin^2\phi)(\sin^2\delta' - \sin^2\phi), \end{aligned}$$

then

$$\begin{aligned} \sin^2\delta &= \mathrm{sn}^2\omega a, \\ \sin^2\delta' &= \mathrm{sn}^2\omega^2 a, \end{aligned}$$

where a has the value in the preceding article § 141, given by

$$\mathrm{cn}\, a = \frac{r^2+\alpha+1}{r^2-\alpha-1},$$

and ω, ω^2 are the imaginary cube roots of unity, so that

$$\begin{aligned} \omega &= -\tfrac{1}{2} + i\tfrac{1}{2}\sqrt{3} = e^{\frac{2}{3}i\pi}, \\ \omega^2 &= -\tfrac{1}{2} - i\tfrac{1}{2}\sqrt{3} = e^{\frac{4}{3}i\pi}. \end{aligned}$$

Then the integral V' is of the form

$$\begin{aligned} V' &= \int \frac{E}{\sin^2\delta - \sin^2\phi}\cdot\frac{d\phi}{\Delta\phi} + \int \frac{F}{\sin^2\delta' - \sin^2\phi}\cdot\frac{d\phi}{\Delta\phi} \\ &= \int \frac{E\,du}{\mathrm{sn}^2\omega a - \mathrm{sn}^2 u} + \int \frac{F\,du}{\mathrm{sn}^2\omega^2 a - \mathrm{sn}^2 u}, \end{aligned}$$

if

$$\frac{d\phi}{\Delta\phi} = du;$$

and therefore V' can be expressed by Jacobi's functions.

It is instructive, however, to show how these results follow from the methods of Legendre, who abandons the problem at its most interesting stage.

From the above known values of ν and θ in terms of α, Legendre says it is necessary to find the value of the parameters m and m', according to his notation; and then (§§ 119, 120)

$$h = \frac{b^2\nu}{\nu^2 + 2c^2\nu\cos\theta + c^2}$$

$$= \frac{(2r^2+3)\{\beta^2 + 3 - r^2(2-\alpha)\}}{\beta(\alpha^2+2\alpha+4)},$$

$$H = \sqrt{(1 + 2h\cos\theta + h^2)}$$

$$= \frac{(\alpha+1)\{\beta^2 + 3 + r^2(2-\alpha)\}}{\beta(\alpha^2+2\alpha+4)},$$

$$\frac{\zeta}{\nu} = H - h = \sin\lambda$$

$$= \frac{\alpha^3 - 3\alpha^2 + 4 - 3r^2\alpha^2}{\beta(\alpha^2+2\alpha+4)};$$

$$m = -c^2\sin^2\lambda;$$

$$\cos\lambda = \frac{2br^3\alpha(2-\alpha)\sqrt{(1+\alpha)}}{\beta(\alpha^2+2\alpha+4)}.$$

But, with our notation,

$$\mathrm{cn}\,a = \frac{r^2+\alpha+1}{r^2-\alpha-1},$$

or

$$\alpha + 1 = -r^2\frac{1-\mathrm{cn}\,a}{1+\mathrm{cn}\,a};$$

a is therefore the value of u, when $y = -\alpha$.

Therefore

$$\beta = 2r^2\frac{\mathrm{dn}\,a}{1+\mathrm{cn}\,a}.$$

Also

$$\alpha^3 + 1 = (\alpha+1)\beta^2$$

$$= -12r^2\frac{(1-\mathrm{cn}\,a)\,\mathrm{dn}^2 a}{(1+\mathrm{cn}\,a)^3},$$

$$\alpha^3 - 3\alpha^2 + 4 - 3r^2\alpha^2 = \alpha^3 + 1 + 3(1 - \alpha^2 - r^2\alpha^2)$$

$$= -24r^2\frac{1 + \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{3}\,\mathrm{cn}^2 a}{(1+\mathrm{cn}\,a)^3},$$

after reduction.

And $$\beta(\alpha^2+2\alpha+4) = 12r^2 \quad \frac{(1+\mathrm{cn}^2 a)\,\mathrm{dn}\,a}{(1+\mathrm{cn}\,a)^3};$$

therefore
$$\sin\lambda = -\frac{1+\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{3}\,\mathrm{cn}^2 a}{(\frac{1}{2}+\frac{1}{2}\,\mathrm{cn}^2 a)\,\mathrm{dn}\,a}$$
$$= -\frac{1+\frac{1}{2}\sqrt{3}\,\mathrm{sn}^2 a}{(1-\frac{1}{2}\,\mathrm{sn}^2 a)\,\mathrm{dn}\,a}$$
$$= -\frac{1}{\mathrm{dn}\,(a\sqrt{3},\, b)}$$

[by means of the cubic transformation, *vide* Cayley's *Elliptic Functions*, p. 187, putting $\alpha = \frac{1}{2}(\sqrt{3}-1)$]
$$= -\,\mathrm{sn}\,(ia\sqrt{3}+K)$$
$$= \quad \mathrm{sn}\,(ia\sqrt{3}-K).$$

If $\alpha+1$ is positive, the value of a is imaginary; but if $\alpha+1$ is negative, a is real; and
$$m = -\,c^2\sin^2\lambda$$
$$= -\,c^2\,\mathrm{sn}^2\,(ia\sqrt{3}+K),$$
so that, according to Legendre's notation, m should be of the form
$$m = -\,1+b^2\sin^2\mu,$$
where μ is a real angle; and then the other modulus m' will be of the form
$$m' = -\,c^2\sin^2\lambda.$$

On the hypothesis that $\alpha+1$ is positive, a is imaginary; and if we put $ia = a'$, then
$$\frac{\zeta'}{\nu} = -H-h$$
$$= -\frac{\alpha+1+r^2}{\beta}$$
$$= \frac{r^2\dfrac{1-\mathrm{cn}\,a}{1+\mathrm{cn}\,a}-r^2}{2r^2\dfrac{\mathrm{dn}\,a}{1+\mathrm{cn}\,a}}$$
$$= -\frac{\mathrm{cn}\,a}{\mathrm{dn}\,a}$$
$$= -\,\mathrm{sn}\,(a+K)$$
$$= \quad \mathrm{sn}\,(ia'-K).$$

Therefore $\sin\mu = -\frac{\zeta'}{\nu}$

$= \operatorname{sn}(a+K)$,

or $= \operatorname{sn}(ia'+K)$;

while $\sin\lambda = \operatorname{sn}(ia\sqrt{3}-K)$,

or $= \operatorname{sn}(a'\sqrt{3}-K)$.

Then

$$m' = -c^2\left(\frac{\alpha+1+r^2}{\beta}\right)^2 = -1+b^2\sin^2\mu',$$

$$\sin\mu' = \frac{\alpha+1-r^2}{\beta}, \quad \cos\mu' = \frac{2cr\sqrt{(\alpha+1)}}{\beta},$$

and μ is a real angle if $\alpha+1$ is negative, μ' is real if $\alpha+1$ is positive.

Legendre says it is not necessary to form the integral in ζ', m', &c., which is of the circular kind if $\alpha+1$ is negative; but he proceeds, on the supposition that $\alpha+1$ is positive, to compare the parameter

$$m' = -1+b^2\sin^2\mu'$$

with the original parameter

$$n = \frac{(\alpha+1-\nu^2)^2}{4\nu^2(\alpha+1)},$$

which is denoted by $\cot^2\gamma$ (μ' is here used instead of μ, as in Legendre's text, where there is a little confusion)

Then $\cot\gamma = \frac{\alpha+1-r^2}{2r\sqrt{(\alpha+1)}}$,

whence $\tan\frac{1}{2}\gamma = \frac{r}{\sqrt{(\alpha+1)}}$;

also $\tan\mu' = \frac{\alpha+1-r^2}{2cr\sqrt{(\alpha+1)}}$;

therefore $\tan\gamma\tan\mu' = \frac{1}{c}$,

and therefore

$$F(b,\gamma)+F(b,\mu') = F'b.$$

But, with Jacobi's notation,

$$n = -\frac{1}{\operatorname{sn}^2 a} = -\frac{1}{\operatorname{sn}^2 ia'} = \frac{\operatorname{cn}^2(a',b)}{\operatorname{sn}^2(a',b)},$$

so that $\gamma = \operatorname{am}(a',b)$.

And
$$\begin{aligned} m' &= -1 + b^2 \sin^2 \mu' \\ &= -c^2 \sin^2 \mu \\ &= -c^2 \operatorname{sn}^2 (ia' + K) \\ &= -\frac{c^2}{\operatorname{dn}^2 (a', b)} \\ &= -\operatorname{dn}^2 (a' - K', b), \end{aligned}$$

so that
$$\mu' = \operatorname{am} (a' - K', b),$$

giving the previous relation with Jacobi's notation.

With the parameters expressed in terms of elliptic functions of a or a', the values of the coefficients B, f, g, and k can be calculated as in Legendre.

In forming the second integrals

$$\sqrt{(-k)} = c^2 \operatorname{cn} \omega a \operatorname{cn} \omega^2 a \frac{\operatorname{sn} a}{\operatorname{dn} a},$$

or
$$= c^2 \operatorname{cn} \omega a \operatorname{cn} \omega^2 a \frac{\operatorname{sn} i\sqrt{3}a}{\operatorname{dn} i\sqrt{3}a},$$

$$A = \frac{\operatorname{cn} \omega a \operatorname{dn} \omega a}{\sqrt{(-k)} \operatorname{sn} \omega a},$$

$$A' = \frac{\operatorname{cn} \omega^2 a \operatorname{dn} \omega^2 a}{\sqrt{(-k)} \operatorname{sn} \omega^2 a},$$

and
$$B = \frac{\operatorname{cn} a \operatorname{dn} a}{\sqrt{(-k)} \operatorname{sn} a},$$

or
$$= \frac{\operatorname{cn} i\sqrt{3}a \operatorname{dn} i\sqrt{3}a}{\sqrt{(-k)} \operatorname{sn} i\sqrt{3}a},$$

with corresponding values of f and g.

But it is simpler to return to the original integral Z, and to show that if the substitutions $y = 1 + x^2$, $x = r \tan \frac{1}{2}\phi$ are employed, then if the denominator $y^3 + \alpha^3$ in terms of ϕ involve the three factors

$$(\cos\phi - \cos\gamma)(\cos\phi - \cos\delta)(\cos\phi - \cos\delta'),$$

and if Jacobi's notation be employed, so that

$$\cos\phi = \operatorname{cn} u, \quad \cos\gamma = \operatorname{cn} a,$$

then
$$\cos\delta = \operatorname{cn} \omega a, \quad \cos\delta' = \operatorname{cn} \omega^2 a,$$

ω and ω^2 being the imaginary cube roots of unity. This has been developed in an article in vol. XVIII. p. 66.

In the two cases of $\alpha = r^2 - 1$ and $\alpha = -r^2 - 1$, the corresponding values of $\operatorname{cn} a$ are ∞ and 0, and of a are iK' and K; so that in the cases the values of the integrals P and Z are seen to be independent of elliptic functions.

Lastly, the case of $\alpha^3 = -4$ gives, as already mentioned, $a = \frac{2}{3}K$, and the integrals are capable of reduction.

ON CERTAIN DEFINITE INTEGRALS CONNECTED WITH SPHERICAL HARMONICS.

By Percival Frost, M.A.

A GREAT variety of Definite Integrals, in which Zonal and general Spherical Harmonics appear, may be obtained by comparison of the results of calculations made on two different lines in order to obtain certain potentials. These integrals are, I imagine, for the most part new, but the method employed seems to be so prolific of curiosities that some of the readers of the *Quarterly Journal* may be interested in the subject.

I shall denote the general Zonal Harmonic by P_i, and when a particular value α is given to μ by $P_i(\alpha)$, and suppose that the letters used in any proposition have their usual signification. One of the expressions which will be used for the potential of a homogeneous ellipsoid at an external point (x, y, z) will be

$$\pi\rho abc \int_\varepsilon^\infty \frac{1 - \dfrac{x^2}{a^2+s} - \dfrac{y^2}{b^2+s} - \dfrac{z^2}{c^2+s}}{\{(a^2+s)(b^2+s)(c^2+s)\}^{\frac{1}{2}}}\, ds,$$

where
$$\frac{x^2}{a^2+\varepsilon} + \frac{y^2}{b^2+\varepsilon} + \frac{z^2}{c^2+\varepsilon} = 1.$$

1. *Integrals derived from the potential of a prolate spheroidal shell at a point on the axis.*

For any external point (x, y, z)

$$V = 2\pi\rho\lambda ab^2 \int_\varepsilon^\infty \frac{ds}{(a^2+s)^{\frac{1}{2}}(b^2+s)},$$

where a, $a(1+\lambda)$ are the semi-axes of the internal and external surfaces, and

$$\frac{x^2}{a^2+\varepsilon} + \frac{y^2+z^2}{b^2+\varepsilon} = 1.$$

If $y=0$, $z=0$, $x^2=a^2+\varepsilon$, and writing $s+\varepsilon$ for s,

$$V=2\pi\rho\lambda ab^2\int_0^\infty \frac{ds}{(x^2+s)^{\frac{1}{2}}(b^2+s)}.$$

Let $x^2+s=u^2$; therefore

$$V=4\pi\rho\lambda ab^2\int_x^\infty \frac{du}{u^2-a^2e^2}=\frac{2\pi\rho b^2}{e}\log\frac{x+ae}{x-ae}$$

$$=\frac{4\pi\rho\lambda b^2}{e}\left\{\frac{ae}{x}+\ldots+\frac{1}{2i+1}\left(\frac{ae}{x}\right)^{2i+1}+\ldots\right\}\ldots\ldots(1).$$

By direct calculation,

$$V=\int_{-1}^{+1}\int_r^{r(1+\lambda)}\frac{\rho.2\pi r^2}{R}\,dr d\mu=\int_{-1}^{+1}\frac{\rho.2\pi r^2}{R}\,r\lambda d\mu,$$

where $$r^2(1-e^2\mu^2)=b^2,$$

and $$R^2=x^2-2rx\mu+x^2;$$

therefore $$V=2\pi\rho\lambda\int_{-1}^{+1}\frac{r^3d\mu}{x}\left\{1+\ldots+P_i\left(\frac{r}{x}\right)^i+\ldots\right\}$$

$$=4\pi\rho\lambda\int_0^1 r^3d\mu\left(\frac{1}{x}+P_2\frac{r^2}{x^3}+\ldots+P_{2i}\frac{r^{2i}}{x^{2i+1}}+\ldots\right)\ldots..(2);$$

hence, equating the two expressions (1) and (2),

$$\int_0^1 r^{2i+3}P_{2i}d\mu=\frac{ab^2}{2i+1}(ae)^{2i}\ldots\ldots\ldots\ldots\ldots(3),$$

or $$\int_0^1\frac{P_{2i}d\mu}{(1-e^2\mu^2)^{\frac{1}{2}(2i+3)}}=\frac{1}{2i+1}e^{2i}(1-e^2)^{-\frac{1}{2}(2i+1)}\ldots(4).$$

This result is easily tested, when $i=2$ or 4, by means of the formula

$$nQ_{m,n+2}=\left(\frac{a}{b}\right)^n-(m+1-n)\,Q_{m,n},$$

where $$Q_{m,n}=\int_0^1\frac{\mu^m d\mu}{(1-e^2\mu^2)^{\frac{1}{2}n}}.$$

Cor. 1. Since $\dfrac{1}{r^2} = \dfrac{\mu^2}{a^2} + \dfrac{1-\mu^2}{b^2}$,

$$\left(a^3 \frac{d}{da} + b^3 \frac{d}{db}\right) \frac{1}{r^2} = -2 \quad \text{or} \quad Dr = r^3;$$

hence, performing the operation $D \equiv a^3 \dfrac{d}{da} + b^3 \dfrac{d}{db}$ on (3),

$$(2i+3) \int_0^1 P_{2i} r^{2i+5} d\mu = \frac{1}{2i+1} \{(ae)^{2i} D(ab^2) + ib^2 (ae)^{2i-2} D(a^2 - b^2)\},$$

$$D(ab^2) = a^3 b^2 + 2ab^4 \quad \text{and} \quad D(a^2 - b^2) = 2(a^4 - b^4);$$

therefore

$$(2i+3)(2i+1) \int_0^1 P_{2i} r^{2i+5} d\mu = ab^2 (ae)^{2i} \{a^2 + 2b^2 + 2i(a^2 + b^2)\}.$$

Now $(4i+3) \mu P_{2i+1} = (2i+2) P_{2i+2} + (2i+1) P_{2i}$;

therefore $(4i+3) \displaystyle\int_0^1 \mu P_{2i+1} r^{2i+5} d\mu = \frac{2i+2}{2i+3} ab^2 (ae)^{2i+2}$

$$+ \frac{1}{2i+3} ab^2 (ae)^{2i} \{(2i+1) a^2 + (2i+2) b^2\}$$

$$= \frac{ab^2}{2i+3} (4i+3) a^2 (ae)^{2i};$$

therefore $$\int_0^1 \mu P_{2i+1} r^{2i+5} d\mu = \frac{a^3 b^2}{2i+3} (ae)^{2i},$$

$$\int_0^1 \frac{\mu P_{2i+1} d\mu}{(1 - e^2 \mu^2)^{\frac{1}{2}(2i+5)}} = \frac{1}{2i+3} e^{2i} (1 - e^2)^{-\frac{1}{2}(2i+3)} \quad \ldots\ldots\ldots\ldots (5).$$

Cor. 2. Since $\dfrac{dr}{r^3} = \dfrac{e^2 \mu d\mu}{b^2}$,

$$(2i+3) \int_0^1 P_{2i+1} r^{2i+2} dr = a (ae)^{2i+2}$$

$$= \left\{ P_{2i+1} r^{2i+3} - \int \frac{dP_{2i+1}}{d\mu} r^{2i+3} d\mu \right\}_0^1;$$

therefore $$\int_0^1 \frac{dP_{2i+1}}{d\mu} r^{2i+3} d\mu = a^{2i+3}(1-e^{2i+2}).$$

$$= \left\{ P_{2i+1} r^{2i+3} - \int \frac{dP_{2i+1}}{d\mu} r^{2i+3} d\mu \right\}_0^1;$$

and $$\int_0^1 \frac{dP_{2i+1}}{d\mu} r^{2i+3} d\mu = a^{2i+3}(1-e^{2i+2}).$$

It can be shown also that

$$\int_0^1 \mu \frac{dP_{2i}}{d\mu} r^{2i+3} d\mu = a^{2i+3}(1-e^{2i}).$$

Cor. 3. By equating powers of e^m in (4) and e^{m-1} in (5), which may be written

$$\int_0^1 \frac{P_i d\mu}{(1-e^2\mu^2)^{\frac{1}{2}(i+3)}} = \frac{e^i (1-e^2)^{-\frac{1}{2}(i+1)}}{i+1}, \quad i \text{ even},$$

and $$\int_0^1 \frac{\mu P_i d\mu}{(1-e^2\mu^2)^{\frac{1}{2}(i+4)}} = \frac{e^{i-1}}{i+2}(1-e^2)^{-\frac{1}{2}(i+2)}, \quad i \text{ odd},$$

we obtain the values of $\int_0^1 \mu^m P_i d\mu$, given by Ferrers, in *Spherical Harmonics*, p. 26, Art. 16.

2 From the potential of an oblate spheroid at a point on the axis we obtain

$$\int_0^1 P_{2i} r^{2i+3} d\mu = (-)^i \frac{a^2 b}{2i+1} (ae)^{2i},$$

where $$r^2 = \frac{b^2}{1-e^2+e^2\mu^2}.$$

3. The integral $\int_0^1 P_{2i} r^{2i+5} d\mu$ may be obtained by taking a shell bounded by confocal prolate spheroids.

At a point in the axis whose abscissa is x the potential is δV, where

$$V = \rho\pi a b^2 \int_0^\infty \frac{sds}{(x^2+s)^{\frac{3}{2}}(x^2+s-\beta^2)} \quad \text{and} \quad \delta a^2 = \delta b^2 = m;$$

therefore, if $x^2+s=u^{-1}$,

$$\delta V = 2\rho\pi m\left(\frac{b^2}{2a}+a\right)\int_0^{x^{-1}}\frac{1-x^2u^2}{1-a^2e^2u^2}\,du$$

$$= 2\rho\pi m\,\frac{2a^2+b^2}{a}\left\{\frac{1}{1.3}\,\frac{1}{x}+\frac{1}{3.5}\,\frac{a^2e^2}{x^3}+\ldots\right\}.$$

Also, directly, $\delta V=\int_{-1}^{+1}\int_r^{r+\delta r}2\rho\pi\,d\mu\,dr\,\frac{r^2}{R}$,

where $$\frac{\delta r^2}{r^4}=\left(\frac{\mu^2}{a^4}+\frac{1-\mu^2}{b^4}\right)\delta a^2;$$

therefore $$\delta r=\frac{mr}{2p^2}=\frac{mr}{2a^2b^2}(a^2+b^2-r^2);$$

and $$\delta V=\int_{-1}^{+1}\pi\rho d\mu\,\frac{mr^3}{2a^2b^2}(a^2+b^2-r^2)\left(1+P_1\frac{r}{a}+\ldots\right)$$

$$=\frac{2\rho\pi m}{a^2b^2}\int_0^1\frac{(a^2+b^2)\,r^3-r^5}{x}\left\{1+P_2\frac{r^2}{x^2}+\ldots\right\}d\mu;$$

therefore

$$\int_0^1 P_{2i}r^{2i+5}d\mu=\frac{ab^2}{2i+1}(ae)^{2i}(a^2+b^2)-\frac{ab^2}{(2i+1)(2i+3)}(ae)^{2i}(2a^2+b^2)$$

$$=\frac{ab^2(ae)^{2i}}{(2i+1)(2i+3)}\{(2i+1)a^2+(2i+2)b^2\}.$$

4. The potential of a shell bounded by similar prolate spheroids at a point (θ_0, ϕ_0, r_0) is by (1),

$$4\pi\rho\lambda ab^2\left(\frac{1}{r_0}+P_2(\cos\psi_0)\,\frac{(ae)^2}{3r_0^3}+\ldots+P_i(\cos\psi)\,\frac{(ae)^{2i}}{(2i+1)\,r_0^{2i+1}}+\ldots\right\}.$$

and, by direct calculation,

$$4\rho\lambda\int_0^{2\pi}\int_0^1 r^3d\mu\,d\phi\left(\frac{1}{r_0}+P_2(\cos\psi)\frac{r^2}{r_0^3}+\ldots+P_{2i}(\cos\psi)\frac{r^{2i}}{r_0^{2i+1}}+\ldots\right),$$

where $\cos\psi=\mu\mu_0+\sqrt{(1-\mu^2)}\,\sqrt{(1-\mu^2)}\cos(\phi-\phi_0)$;

therefore

$$\int_0^{2\pi}\int_0^1 P_{2i}(\cos\psi)\,r^{2i+3}d\mu\,d\phi=\frac{2\pi ab^2}{2i+1}P_{2i}(\cos\psi_0)(ae)^{2i}.$$

5. I shall now obtain corresponding results for the general case of an ellipsoid.

The potential of a shell bound by similar ellipsoids

$$= V = 2\pi\rho abc\lambda \int_0^\infty \frac{ds}{D},$$

at an external point (x, y, z), where

$$D^2 = (a_1^2 + s)(b_1^2 + s)(c_1^2 + s), \quad \delta a = 4a, \quad \&c.$$

and

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1.$$

Let $a_1^2 + s = -\frac{1}{u^2}$; therefore $ds = -\frac{2du}{u^3}$;

$$V = 4\pi\rho abc\lambda \int_0^{a_1^{-1}} \frac{du}{\{(1-\beta^2u^2)(1-\gamma^2u^2)\}^{\frac{1}{2}}}$$

$$= 4\pi\rho abc\lambda \int_0^{a_1^{-1}} \frac{du}{\{1-(\beta^2+\gamma^2)u^2-\beta^2\gamma^2u^4\}^{\frac{1}{2}}}$$

$$= 4\pi\rho abc\lambda \left\{\frac{1}{a_1} + P_1\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)\frac{\beta\gamma}{3a_1^3} + \ldots \right.$$

$$\left. + P_i\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)\frac{(\beta\gamma)^i}{(2i+1)\,a_1^{2i+1}} + \ldots\right\}.$$

But by direct calculation

$$V = \int_0^{2\pi}\int_{-1}^{+1}\int_r^{(1+\lambda)r} \frac{\rho\, d\phi\, d\mu\, .\, r^2 dr}{(r_1^2 - 2r_1 r\cos\psi + r^2)^{\frac{1}{2}}}$$

$$= \int_{2\pi}^{2\pi}\int_{-1}^{+1} \rho\, d\phi\, d\mu \int_r^{r+\lambda r} \frac{1}{r_1}\left\{r^2 + \ldots + P_{2i}(\cos\psi)\frac{r^{2i+2}}{r_1^{2i}} + \ldots\right\}$$

$$= 2\rho\lambda \int_0^{2\pi}\int_0^1 d\phi\, d\mu \left\{\frac{r^3}{r_1} + P_2(\cos\psi)\frac{r^5}{r_1^3} + \ldots \right.$$

$$\left. + P_{2i}(\cos\psi)\frac{r^{2i+3}}{r_1^{2i+1}} + \ldots\right\}.$$

If $y=0$, $z=0$, then $r_1=x$, $\psi=\theta$, $\cos\psi=\mu$; and equating coefficients of $\frac{1}{x^{2i+1}}$,

$$\int_0^{2\pi}\int_0^1 d\phi\, d\mu\,.\,P_{2i}r^{2i+3}=\frac{2\pi abc}{2i+1}P_i\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)(\beta\gamma)^i\ldots(1).$$

6. To test this result by obtaining the definite integral directly in the particular case where $i=1$ and 2, we have to evaluate

$$\int_0^{2\pi}\int_0^1 d\mu\, d\phi\,.\,\frac{35\mu^4-30\mu^2+3}{8}r^7$$

$$=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\,(35\mu^4-30\mu^2+3)\,r^7,$$

$$\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\,.\,r^3=\int_0^{\frac{1}{2}\pi}\int_0^1\frac{d\mu\, a\phi}{\left\{\frac{\mu^2}{a^2}+(1-\mu^2)\left(\frac{\cos^2\phi}{b^2}+\frac{\sin^2\phi}{c^2}\right)\right\}^{\frac{3}{2}}}$$

$$=\int_0^{\frac{1}{2}\pi}\frac{a\,d\phi}{\frac{\cos^2\phi}{b^2}+\frac{\sin^2\phi}{c^2}}=\tfrac{1}{2}\pi abc\ldots\ldots\ldots\ldots(1).$$

Performing the operation $a^3\frac{d}{da}+b^3\frac{d}{db}+c^3\frac{d}{dc}$ on both sides,

$$3\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\, r^2=\tfrac{1}{2}\pi abc\,(a^2+b^2+c^2)\ldots\ldots(2),$$

the operation $a^3\frac{d}{da}$ on (1) gives

$$3\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\,\mu^2r^5=\tfrac{1}{2}\pi a^3bc\ldots\ldots\ldots(3);$$

whence $$\int_0^{2\pi}\int_0^1 d\mu\, d\phi P_2r^5=\pi a^3bc-\tfrac{1}{3}\pi abc\,(a^2+b^2+c^2)$$

$$=\frac{\pi abc}{3}(\beta^2+\gamma^2)=\frac{2\pi abc}{3}P_1\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)\beta\gamma.$$

Similarly by (2),

$$3.5\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\, r^7 = \tfrac{1}{2}\pi\{3a^5bc + a^3bc\,(b^2+c^2)+\ldots\}$$
$$= \tfrac{1}{2}\pi abc\{2\,(a^4+b^4+c^4)+(a^2+b^2+c^2)^2\},$$

and $3.5\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\, \mu^2 r^7 = \tfrac{1}{2}\pi\{3a^5bc + a^3bc\,(b^2+c^2)\}$;

also by (3), $3.5\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\, \mu^4 r^7 = \tfrac{1}{2}\pi\, 3a^5bc$;

therefore $\int_0^{\frac{1}{2}\pi}\int_0^1 d\mu\, d\phi\, P_4 r^7 = \tfrac{1}{4}\pi abc\,[7a^4 - 2\{3a^4 + a^2(b^2+c^2)\}$
$+ \tfrac{1}{5}\{2\,(a^4+b^4+c^4)+(a^2+b^2+c^2)^2\}]$
$= \tfrac{1}{2^0}\pi abc\{8a^4 + 3\,(b^4+c^4) - 8a^2(b^2+c^2)+2b^2c^2\}$
$= \dfrac{2\pi abc}{8.5}\{3\,(b^2-c^2)^2 + 8\,(a^2-b^2)\,(a^2-c^2)\}$
$= \dfrac{2\pi abc}{8.5}\{3\,(\beta^2+\gamma^2)^2 - 4\beta^2\gamma^2\}$
$= \dfrac{2\pi abc}{5} P_2\left(\dfrac{\beta^2+\gamma^2}{2\beta\gamma}\right)(\beta\gamma)^2.$

7. For a shell bounded by confocal ellipsoids

$$\delta V = \int_0^{2\pi}\int_{-1}^{+1}\rho\, d\mu\, d\phi \int_r^{r+\delta r}\frac{r^2 dr}{D}$$
$$= \int_0^{2\pi}\int_0^1 2\rho d\mu d\phi\,\frac{r^2 dr}{x}\left(1 + P_2\frac{r^2}{x^2}+\ldots\right)$$
$$\delta\frac{1}{r^3} = -\frac{\mu^2\delta a^2}{a^4} - (1-\mu^2)\left(\frac{\cos^2\phi\,\delta b^2}{b^4} + \frac{\sin^2\phi\,\delta c^2}{c^4}\right),$$
$$\delta a^2 = \delta b^2 = \delta c^2,$$
$$\frac{\delta r}{r^3} = a\delta a\left\{\frac{\mu^2}{a^4} + (1-\mu^2)\left(\frac{\cos^2\phi}{b^4} + \frac{\sin^2\phi}{c^4}\right)\right\},$$
$$\delta V = \int_0^{2\pi}\int_0^1 2a\rho\, d\mu\, d\phi\,\delta a\frac{r^5}{x}\left(1 + P_2\frac{r^2}{x^2}+\ldots\right)$$
$$\times\left\{\frac{\mu^2}{a^4} + (1-\mu^2)\left(\frac{\cos^2\phi}{a^4} + \frac{\sin^2\phi}{b^4}\right)\right\},$$

and $\delta V = 4\pi\rho\delta a\left(bc + \frac{ac^2}{b} + \frac{a^2b}{c}\right)$

$$\left\{\frac{1}{1.3}\cdot\frac{1}{x} + \ldots + \frac{1}{(2i+1)(2i+3)}P_i\left(\frac{\beta^2+\gamma}{2\beta\gamma}\right)(\beta\gamma)^i\frac{1}{x^{2i+1}};\right.$$

therefore $\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, r^{2i+5}\left\{\frac{\mu^2}{a^4} + (1-\mu^2)\left(\frac{\cos^2\phi}{b^4} + \frac{\sin^2\phi}{c^4}\right)\right\}$

$$= \frac{2\pi abc}{(2i+1)(2i+3)}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)P_i\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)(\beta\gamma)^i.$$

This may also be obtained from (1) of Art. 4, by performing the operation $\frac{1}{a}\frac{d}{da} + \frac{1}{b}\frac{d}{db} + \frac{1}{c}\frac{d}{db}$ on both sides,

$$(2i+3)\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, P_{2i}r^{2i+5}\left\{\frac{\mu^2}{a^4} + (1-\mu^2)\left(\frac{\cos^2\phi}{b^2} + \frac{\sin^2\phi}{c^2}\right)\right\}$$

$$= \frac{2\pi(\beta\gamma)^i}{2i+1}P_i\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)\left(\frac{1}{a}\frac{d}{da} + \ldots\right)abc,$$

since $\left(\frac{1}{a}\frac{d}{da} + \frac{1}{b}\frac{d}{db} + \frac{1}{c}\frac{d}{dc}\right)\beta$ or $\gamma = 0$;

therefore $\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, P_{2i}\frac{r^{2i+3}}{p^2}$

$$= \frac{2\pi abc}{(2i+1)(2i+3)}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)P_i\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)(\beta\gamma)^i.$$

8. To find $\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, P_{2i}r^{2i+5}$.

Since $\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, P_{2i}r^{2i+3} = \frac{2\pi abc}{2i+1}P_i\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right)(\beta\gamma)^i$,

performing the operation

$$D \equiv a^3\frac{d}{da} + b^3\frac{d}{db} + c^3\frac{d}{dc},$$

since $D(\beta) = \frac{a^4-b^4}{\sqrt{(a^2-b^2)}} = (a^2+b^2)\beta$,

$$D(\beta\gamma) = (2a^2+b^2+c^2)\beta\gamma,$$

$$D\left(\frac{\beta}{\gamma}\right) = \frac{(b^2-c^2)\beta\gamma}{\gamma^2} = (\gamma^2-\beta^2)\frac{\beta}{\gamma};$$

therefore $D\dfrac{\beta^2+\gamma^2}{2\beta\gamma} = \frac{1}{2}(\gamma^2-\beta^2)\left(\dfrac{\beta}{\gamma}-\dfrac{\gamma}{\beta}\right) = -\dfrac{(\gamma^2-\beta^2)^2}{2\beta\gamma}$;

therefore $(2i+3)\displaystyle\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, P_{2i} r^{2i+3}$

$$= \frac{2\pi abc}{ai+1}\{(a^2+b^2+c^2)\,P_i(v)\,(\beta\gamma)^i + i\,(2a^2+b^2+c^2)\,P_i(v)\,(\beta\gamma)^i - \tfrac{1}{2}(\gamma^2-\beta^2)^2\frac{d}{dv}P_i(v)\,(\beta\gamma)^{i-1}\};$$

therefore $$\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, P_{2i} r^{2i+3} = \frac{2\pi abc}{(2i+1)\,(2i+3)}$$

$$\times\{(2i+1)\,a^2+(i+1)\,(b^2+c^2)\,P_i(v)\,(\beta\gamma)^i - \tfrac{1}{2}(\gamma^2-\beta)^2\frac{d}{dv}P_i(\beta\gamma)^{i-1}\}.$$

Cor. From the results of Arts. 5 and 7,

$$\int_0^{2\pi}\int_0^1 d\mu\, d\phi\, \mu P_{2i+1} r^{2i+3} = \frac{2\pi abc}{(4i+3)\,(2i+3)}\,[(2i+2)P_{i+1}(v)(\beta\gamma)^{i+1}$$

$$+\{(2i+1)\,a^2+(i+1)\,b^2\}P_i(v)\,(\beta\gamma)^i - \tfrac{1}{2}(\gamma^2-\beta^2)^2\frac{d}{dv}P_i(v)(\beta\gamma)^{i-1}],$$

which reduces to $\dfrac{2\pi a^3b^2}{2i+3}(ae)^{2i}$ when $b=c$ and $\beta=\gamma=ae$.

9. The potential of a homogeneous prolate spheroid at a point on the axis, whose distance from the focus is x being V,

$$V = \pi\rho ab^2\int_0^\infty \frac{\left\{1-\dfrac{(x-ae)}{a_1^2+s}\right\}ds}{(a_1^2+s)^{\frac{1}{2}}\,(b_1^2+s)},$$

where $$x-ae=a_1,\quad a^2e^2=a_1^2-b_1^2,$$

$$= \pi\rho ab^2\int_0^\infty \frac{sds}{(a_1^2+s)^{\frac{3}{2}}\,(b_1^2+s)};$$

and if $a_1^2+s=\dfrac{1}{u^2}$, $\qquad ds=-\dfrac{2du}{u^3}$,

$$V = 2\pi\rho ab^2 \int_0^{a_1^{-1}} \frac{1 - a_1^2 u^2}{1 - a^2e^2u^2}\, du$$

$$= 2\pi\rho \frac{ab^2}{a^2e^2} \int_0^{a_1^{-1}} \frac{a^2e^2 - a_1^2 + a_1^2(1 - a^2e^2u^2)}{1 - a^2e^2u^2}\, du$$

$$= \frac{2\pi\rho ab^2}{a^2e^2} \left(a_1 - \frac{b_1^2}{2ae} \log \frac{a_1 + ae}{a_1 - ae} \right)$$

$$= \frac{2\pi\rho ab^2}{a^2e^2} \left(x - ae - \frac{x^2 - 2aex}{2ae} \log \frac{x}{x - 2ae} \right)$$

$$= \frac{2\pi\rho ab^2}{a^2e^2} \left[x - ae - (x - 2ae) \left\{ 1 + \frac{2ae}{2x} + \tfrac{1}{3} \left(\frac{2ae}{x} \right)^2 + \ldots \right\} \right]$$

$$= 8\pi\rho ab^2 \left(\frac{1}{2.3x} + \frac{2ae}{3.4x^2} + \frac{(2ae)^2}{4.5x^3} + \ldots \right).$$

Hence the potential of a shell bounded by similar spheroids having one focus common and the axes in the same directions, the lengths of which are as $1 : 1 + \lambda$, is

$$\delta V = 8\pi\rho ab^2.3\lambda \left(\frac{1}{2.3x} + \frac{2ae}{3.4x^2} + \frac{(2ae)^2}{4.5x^3} + \ldots \right)$$

$$+ 8\pi\rho ab^2.\lambda \left(\frac{2ae}{3.4x^2} + \frac{2\,(2ae)^2}{4.5x^3} + \frac{3\,(2ae)^3}{5.6x^4} + \ldots \right)$$

$$= 8\pi\rho ab^2\lambda \left(\frac{1}{2x} + \frac{2ae}{3x^2} + \frac{(2ae)^2}{3x^3} + \frac{(2ae)^3}{5x^4} + \ldots \right).$$

Working directly,

$$\delta V = \int_{-1}^{+1} \int_r^{r+\lambda r} \frac{2\pi\rho r^2 d\mu}{(x^2 - 2xr\mu + r^2)^{\frac{1}{2}}}$$

$$= \int_{-1}^{+1} \frac{2\pi\rho r^2.\lambda r}{x} \left(1 + P_1 \frac{r}{x} + P_2 \frac{r^2}{x^2} + \ldots \right);$$

therefore, equating coefficients of $\frac{1}{x^{i+1}}$,

$$\int_{-1}^{+1} P_i r^{i+3} d\mu = \frac{4ab^2}{i+2} (2ae)^i,$$

or

$$\int_{-1}^{+1} \frac{P_i d\mu}{(1 - e\cos\theta)^{i+2}} = \frac{2^{i+2}}{i+2} e^i (1 - e^2)^{-(i+2)}.$$

10. If S be a point within a sphere whose centre is C, and SC be produced to X external to the sphere, and $SX=x$, $SC=c$, the potential of the sphere at X is $\frac{4\pi\rho a^3}{3(x-c)}$; and, calculated directly, it is $\int_{-1}^{+1}\int_0^r \frac{2\pi\rho r^2 dr d\mu}{R}$,

when $R^2=x^2-2rx\mu+r^2$, and $r^2-2cr\mu+c^2=a^2$;

therefore $\frac{2a^3}{3x}\left(1+\frac{c}{x}+\frac{c^2}{x^2}+\ldots\right)=\int_{-1}^{+1}\int_0^r \frac{d\mu}{x}\left(r^2+\frac{r^3}{x}P_1+\ldots\right)dr$

$$=\int_{-1}^{+1}\frac{d\mu}{x}\left(\frac{r^3}{3}+\frac{r^4}{4x}P_1+\ldots\right);$$

and, equating coefficients of $\frac{1}{x^{i+1}}$,

$$\int_{-1}^{+1} d\mu P_i\{\sqrt{(a^2-c^2+\delta\mu^2)}+c\mu\}^{i+3}=\tfrac{2}{3}(i+3)a^3c^i$$

$$=\int_0^1 d\mu P_i\{\sqrt{(a^2-c^2+c^2\mu^2)}+c\mu\}^{i+3}-\{c\mu-\sqrt{(a^2-c^2+c^2\mu^2)}^{i+3}\}.$$

Cor. 1. If $a=c$, $\quad\int_0^{+1} d\mu P_i(2a\mu)^{i+3}=\tfrac{2}{3}(i+3)a^{i+3}$,

which gives a curious transformation of the expression given by Ferrers of $\int_0^1 \mu^m P_i d\mu$ when $m=i+3$,

viz. $\frac{5.7\ldots(i+2)}{(2i+4)(2i+2)\ldots(i+5)}$ when i is odd,

and $\frac{5.7\ldots(i+3)}{(2i+4)(2i+2)\ldots(i+4)}$ when i is even,

are both reducible to $\frac{i+3}{3.2^{i+2}}$.

I may observe that $\int_0^1 P_i\mu^m d\mu$ may be put into the forms

$$\frac{1}{2^{\frac{1}{2}(m-i)}}\,\frac{\frac{1}{2}(m-i+2)\ldots(m-1)\,m}{3.5\ldots(m+i+1)}$$ when $m+i$ is even,

$$\frac{1}{2^{\frac{1}{2}(m+i+1)}}\,\frac{\frac{1}{2}(m+i+3)\ldots(m-1)\,m}{3.5\ldots(m-i)}$$ when $m+i$ is odd,

of which the above is a particular case, viz. $m=i+3$.

Cor. 2 If we write $\sqrt{(a^2 - c^2 + c^2\mu^2)} = u$,

$$\int_0^1 d\mu P_i \left((i+3)(c\mu)^{i+2} u + \frac{(i+3)(i+2)(i+1)}{3!} u^3 + \ldots \right) = \frac{i+3}{3} a^3 c^i,$$

the last terms in the series being

$$\frac{(i+3)(i+2)}{2!} c^2\mu^2 u^{i+1} + u^{i+3}, \text{ when } i \text{ is even};$$

and $\dfrac{(i+3)(i+2)(i+1)}{3!} c^3\mu^3 u^i + (i+3) c\mu u^{i+2}$, when i is odd.

Now $$\int_0^1 \mu^s u^n d\mu = \frac{a^n}{s+1} - \frac{n}{s+1} \int_0^1 c^2 \mu^{s+2} u^{n-2} d\mu,$$

$$\int_0^1 \mu^{s+2} u^{n-2} d\mu = \frac{a^{n-2}}{s+3} - \frac{n^{-2}}{s+3} \int_0^1 c^2 u^{s+4} u^{n-4} d\mu,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

n odd, $$\int_0^1 \mu^{s+n-3} u^3 d\mu = \frac{a^3}{s+n-2} - \frac{3}{s+n-2} \int_0^1 c^2 \mu^{s+n-1} u d\mu;$$

therefore, omitting the terms of lower powers than c^{n-1},

$$\int_0^1 \mu^s u^n d\mu = (-1)^{\frac{1}{2}(n-1)} \frac{n(n-2)\ldots 5.3}{(s+1)(s+3)\ldots(s+n-2)} c^{n-1} \int_0^1 \mu^{s+n-1} u d\mu.$$

Let $$P_i = A_i \mu^i + A_{i-2} \mu^{i+2} + \ldots;$$

therefore, when i is even,

$$\int_0^1 d\mu (A_i \mu^i + A_{i-2} \mu^{i-2} + \ldots)(u^{i+3} + \frac{(i+3)(i+2)}{2} c^2 \mu^2 u^{i+1} \ldots = \tfrac{1}{3}(i+3) a^3 c^i.$$

The terms in the coefficient of A_i involving powers above c^i are

$$(-1)^{\frac{1}{2}(i+2)} \frac{(i+3)(i+1)\ldots 3}{(s+1)(s+3)\ldots(s+i+1)} \left\{ 1 - \frac{(i+3)(i+2)}{2} \frac{s+1}{i+3} \right.$$

$$\left. + \frac{(i+3)(i+2)(i+1)^i}{4!} \frac{(s+1)(s+3)}{(i+3)(i+1)} - \ldots \right\} c^{i+2} \int_0^1 \mu^{s+i+2} u d\mu$$

$$= C_i c^{i+2} \int_0^1 \mu^{s+i+2} u \, d\mu;$$

therefore, for all values of c,

$$\int_0^1 (A_i C_i \mu^{2i+2} + A_{i-2} C_{i-2} \mu^{2i} + \ldots)\, u d\mu = 0,$$

or $$1 - \frac{(i+2)(s+1)}{2} + \frac{(i+2)\,i(s+1)(s+3)}{4!} - \ldots = 0,$$

for all even values of i and s, s not greater than i.

Similarly, when i and s are both odd,

$$1 - \frac{(i+1)(s+2)}{3!} + \frac{(i+1)(i-1)(s+2)(s+4)}{5!} - \ldots = 0;$$

therefore for all integral values when s is not greater than n,

$$1 - \frac{(n+1)(2s+1)}{2!}2 + \frac{(n+1)n(2s+1)(2s+3)}{4!}2^2 - \ldots = 0,$$

and when s is not greater than $n+1$,

$$1 - \frac{(n+1)(2s+1)}{3!}2 + \frac{(n+1)n(2s+1)(2s+3)}{5!}2^2 - \ldots = 0.$$

These series are respectively the values, when $x = 1$, of

$$\frac{2^s}{1.3\ldots(2s-1)}\left(\frac{d}{dx}\right)^s \{x^{\frac{1}{2}(2s-1)}(1-x)^{n+1}\}$$

and $$\frac{2^{s-1}}{3.5\ldots(2s-1)}\left(\frac{d}{dx}\right)^{s-1} \{x^{\frac{1}{2}(2s-1)}(1-x)^{n+1}\},$$

which are obviously zero, with the restrictions given.

Cor. 3. If we equate the lower powers up to c^i, we have equations sufficient to determine $A_i A_{i-2} \ldots$.

Thus, when $i = 2$,

$$\int_0^1 (A_2\mu^2 + A_0)(u^5 + 10c^2\mu^2 u^3 + 5c^4\mu^4 u)\, \tfrac{5}{3}a^3c^2,$$

$$A_2\left(\frac{a^5}{3} - \frac{5}{3.5}a^3c^3 + \tfrac{10}{3}a^3c^2\right)$$

$$+ A_0\left(\frac{a^5}{1} - \frac{5}{1.3}a^3c^2 + \tfrac{10}{5}a^3c^2\right) = \tfrac{5}{3}a^3c^2,$$

$$A_2 + 3A_0 = 0, \quad A_2 + A_0 = 1;$$

therefore $$A_2 = \tfrac{3}{2}, \quad A_0 = -\tfrac{1}{2}.$$

Also, when $i=3$,

$$\int_0^1 (A_3\mu^3 + A_1\mu)\,(c\mu u^5 + \frac{5.4}{3!}\,c^3\mu^3u^3 + c^5\mu^5u) = \tfrac{1}{3}a^3c^3,$$

therefore
$$A_3\left(\frac{a^5}{5} - \frac{5}{5.7}\,a^3c^2 + \frac{10}{3}\,\frac{a^3c^2}{7}\right)$$
$$+ A_1\left(\frac{a^5}{5} - \frac{5}{3.5}\,a^3c^2 + \frac{10}{3}\,\frac{a^3c^2}{5}\right) = \tfrac{1}{3}a^3c^2,$$
$$\tfrac{3}{5}A_3 + \tfrac{1}{3}A_1 = 0 \quad \text{and} \quad A_3 + A_1 = 1;$$
therefore
$$A_3 = \tfrac{5}{2}, \quad A_1 = -\tfrac{3}{2}.$$

11. To show, by the methods employed in this paper, that

$$\int_0^{2\pi}\int_{-1}^{+1} d\mu\,d\phi\; Y_iP_i\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\,(\cos\phi - \phi')\} = \frac{4\pi}{2i+1}\,Y_i',$$

where Y_i is a surface harmonic of degree i, and Y_i' is its value at a point (R, θ', ϕ').

Let a spherical shell, whose internal and external radii are a and $a+\delta a$, have a density at every point which is a surface harmonic ρY_i, then, as shown in Ferrers' *General Spherical Harmonics*, Art. 3, the potential at an external point $P(R, \theta', \phi')$

$$= \frac{4\pi a^2\delta a}{2i+1}\,\rho Y_i'\,\frac{a^i}{R^{i+1}}.$$

But, by direct calculation, the potential at P

$$= \int_0^{2\pi}\int_{-1}^{+1} d\mu\,d\phi\,\rho\,Y_i \int_a^{a+\delta a} \frac{r^2dr}{(R^2 - 2aR\cos\psi + a^2)^{\frac{1}{2}}}$$
$$= \int_0^{2\pi}\int_{-1}^{+1} d\mu\,d\phi\,\rho\,Y_i\,\frac{a^2\delta a}{R}\left\{1 + \ldots + P_i(\cos\psi)\left(\frac{a}{R}\right)^i + \ldots\right\};$$

therefore, equating coefficients of powers of $\frac{1}{R}$,

$$\int_0^{2\pi}\int_{-1}^{+1} d\mu\,d\phi\,Y_iP_j(\cos\psi) = 0, \text{ if } i \text{ and } j \text{ are unequal,}$$

and
$$\int_0^{2\pi}\int_{-1}^{+1} d\mu\,d\phi\,Y_iP_i(\cos\psi) = \frac{4\pi}{2i+1}\,Y_i'.$$

ON THE SOLUTION OF THE GENERAL EQUATION OF THE FOURTH DEGREE.

By A. Lodge, St. John's College, Oxford.

REFERRING to the paper by M. Legoux, which appeared in this Journal in October last, on an application of the special form of determinant, previously discussed by Mr. Glaisher and Mr. Scott in this Journal in 1879 and 1880, I have, by means of this form of determinant, been led to an extremely simple solution of the general equation of the fourth degree, which appears to have been previously overlooked.

Transforming the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \quad \ldots\ldots\ldots\ldots (1)$$

so as to lose its second term, by putting $y = ax + b$, we obtain

$$y^4 + 6Hy^2 + 4Gy + F = 0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Let us now identify this expression with the determinant

$$\begin{vmatrix} y, & p, & q, & r \\ r, & y, & p, & q \\ q, & r, & y, & p \\ p, & q, & r, & y \end{vmatrix}.$$

Splitting this determinant into factors in the manner shown by Mr. Scott, it reduces to

$$\begin{vmatrix} y+q, & p+r \\ p+r, & y+q \end{vmatrix} \times \begin{vmatrix} y-q, & p-r \\ r-p, & y-q \end{vmatrix}$$

$$= \{(y+q)^2 - (p+r)^2\}\{(y-q)^2 + (p-r)^2\}$$

$$= (y^2 + q^2 - 2pr)^2 - (2qy - p^2 - r^2)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$= y^4 - 2y^2(q^2 + 2pr) + 4qy(p^2 + r^2) + (q^2 - 2pr)^2 - (p^2 + r^2)^2 \ldots (4).$$

Comparing coefficients in (4) and (2), we obtain

$$\left.\begin{aligned} q^2 + 2pr &= -3H \\ q(p^2 + r^2) &= G \\ (q^2 - 2pr)^2 - (p^2 + r^2)^2 &= F \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5),$$

whence $$4q^6 + 12Hq^4 + (9H^2 - F)q^2 - G^2 = 0 \quad \ldots\ldots\ldots (6).$$

By making use of the values

$$F = a^2 I - 3H^2, \quad G^2 = a^2 (HI - aJ) - 4H^3,$$

where I, J are the two invariants of the quartic, this equation may be written in either of the forms

$$4q^6 + 12Hq^4 + (12H^2 - a^2 I)\, q^2 - G^2 = 0 \ldots\ldots\ldots (6')$$

or

$$4\left(\frac{q^2 + H}{a}\right)^3 - I\frac{q^2 + H}{a} + J = 0 \ldots\ldots\ldots\ldots (6'').$$

Again, substituting from (5) in (3), we obtain

$$(y^2 + 3H + 2q^2)^2 - \left(2qy - \frac{G}{q}\right)^2 = 0;$$

therefore

$$\left\{(y+q)^2 + 3H + q^2 - \frac{G}{q}\right\}\left\{(y-q)^2 + 3H + q^2 + \frac{G}{q}\right\} = 0;$$

that is

$$\left\{(ax + b + q)^2 + 3H + q^2 - \frac{G}{q}\right\} \times \left\{(ax + b - q)^2 + 3H + q^2 + \frac{G}{q}\right\} = 0 \quad \ldots\ldots (7).$$

On finding any value of q^2 from (6) and substituting it in any of the equations (7), we thus split the biquadratic into two quadratic factors, each of which can be at once solved for the roots.

I have given the above method of arriving at equations (7) and the reducing cubic (6), as it was the way in which I was myself led to them; but the first of equations (7) is so simple, and, when once arrived at, so self-evident, that it might have been sufficient to have written it down immediately after (2).

On expanding it, and subtracting (2) from it, we arrive at once at the reducing cubic (6).

The roots of the biquadratic are most intimately connected with those of the reducing cubic. Of these latter, one is positive, and the other two are either both positive, both negative, or both imaginary. Moreover, in the first case two or all of the roots may be equal, and in the second case the two negative roots may be equal. Also any one or more of the roots may be zero. To each of these cases corresponds some special peculiarity in the roots of the biquadratic.

To investigate the general conditions for real or imaginary roots of the biquadratic, let α, β be the roots corresponding

to the first factor of (7), and γ, δ the roots corresponding to the second factor.

Then $$\tfrac{1}{4}a^2(\alpha-\beta)^2 = \frac{G}{q} - (q^2+3H)$$

and $$\tfrac{1}{4}a^2(\gamma-\delta)^2 = -\frac{G}{q} - (q^2+3H).$$

Hence all the roots will be real, or all imaginary, if

$$(q^2+3H)^2 - \frac{G^2}{q^2} \text{ is positive;}$$

and two of the roots will be real and two imaginary, if

$$(q^2+3H)^2 - \frac{G^2}{q^2} \text{ is negative.}$$

Taking the first case, and substituting from equation (6), we have

$$(q^2+3H)^2 > 4q^4 + 12Hq^2 + 12H^2 - a^2I;$$

therefore $$a^2I > 3(q^2+H)^2;$$

therefore $$I > 3\phi^2, \text{ where } a\phi = q^2+H.$$

But (from 6″) $$4\phi^3 - I\phi + J = 0;$$

therefore $$\phi^2(4\phi^2 - I)^2 = J^2;$$

therefore $$I^3 - 27J^2 \text{ is positive} \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Similarly, in the second case, that is, when there are two real and two imaginary roots,

$$I^3 - 27J^2 \text{ is negative} \ldots\ldots\ldots\ldots\ldots\ldots (9);$$

and if two of the roots are equal,

$$I^3 - 27J^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

(The equal roots must of course be real, unless the quartic is a perfect square, when they can be either real or imaginary.)

But $I^3 - 27J^2$ is the discriminant (with its sign changed) of the reducing cubic; therefore, if it is positive, all the roots of the cubic are real; if it is negative, two roots are imaginary; and if it is zero, two roots are equal.

Hence, if all the roots of the reducing cubic are real, the roots of the biquadratic are either all real or all imaginary; if the cubic has two imaginary roots, the biquadratic has two roots real and two imaginary; and if the reducing cubic has two equal roots, so also has the biquadratic.

Further, if all the values of q^2 are not only real, but positive, all the roots of the biquadratic must be real, for all its quadratic factors will have real coefficients, which can

only happen when the linear factors involve only real quantities. There will be no negative root of (6) if its coefficients are alternately positive and negative; and therefore the necessary and sufficient conditions for all the roots of the biquadratic to be real are

$$\left.\begin{array}{r} H \text{ negative}.....(\text{i}) \\ 12H^2 - a^2I \text{ positive} .....(\text{ii}) \\ I^3 - 27J^2 \text{ positive} .....(\text{iii}) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (11),$$

these conditions simply expressing that the roots of the reducing cubic (6) are all real and positive.

Now the conditions for all the roots of the biquadratic to be real are found, by applying Sturm's theorem, to be

$$\left.\begin{array}{r} H \text{ negative}.....(\text{i}) \\ 2HI - 3aJ \text{ negative}.....(\text{ii}) \\ I^3 - 27J^2 \text{ positive} .....(\text{iii}) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (12);$$

and it is important to reconcile the apparent discrepancy between these two sets of conditions.

We have the identity

$$H(12H^2 - a^2I) + 3G^2 \equiv a^2(2HI - 3aJ),$$

since $$G^2 + 4H^3 \equiv a^2(HI - aJ);$$

therefore {12 (ii)} expresses not only that $12H^2 = a^2I$ must be positive, but also that it must be greater than $\frac{3G^2}{-H}$.

And it is readily proveable that this condition must be fulfilled if the roots of the cubic (6) are all positive; for, let l, m, n be these roots, then

$$l^3 + m^3 + n^3 - 3lmn \equiv (l + m + n)(l^2 + m^2 + n^2 - mn - nl - lm);$$

therefore $$(l + m + n)(mn + nl + lm) - 9lmn$$

$$\equiv (l + m + n)(l^2 + m^2 + n^2) - (l^3 + m^3 + n^3) - 6lmn$$

$$\equiv l(m - n)^2 + m(n - l)^2 + n(l - m)^2,$$

which is positive if l, m, n are all positive; therefore

$$-H(12H^2 - a^2I) - 3G^2 \text{ must be positive;}$$

that is, $$2HI - 3aJ \text{ must be negative.}$$

But this condition is implicitly contained in those of (11), as can be seen from inspection of the easily verified identity

$$a^2(I^3 - 27J^2)$$

$$\equiv 12HI(2HI - 3aJ) - 3(2HI - 3aJ)^2 - I^2(12H^2 - a^2I).$$

Hence either series of conditions (11) or (12) may be employed at pleasure, but in practice those of (11) would generally, if not always, be most easy of application.

If any of the values of q^2 are negative, there will in general be no real root of the biquadratic, as can at once be seen by writing (7) in the form

$$(y^2 + 3H + 2q^2)^2 - q^2\left(2y - \frac{G}{q^2}\right)^2,$$

which, for a negative value of q^2, is the sum of two squares, and cannot therefore vanish for a real value of y, except in the special case in which both squares vanish simultaneously.

This special case is that of equal roots, for, eliminating q from the two equations

$$2y - \frac{G}{q^2} = 0,$$

$$y^2 + 3H + 2q^2 = 0,$$

we obtain $$y^3 + 3Hy + G = 0.$$

But this is the first derived function from (2), and can only vanish with it when (2) has equal roots.

(In this case we know that (6) will also have equal roots, a fact which can also be established by eliminating y from the above simultaneous equations.)

If the cubic has all its roots equal, the biquadratic will have three equal roots. For in this case $I = 0$ and $J = 0$, and (6) reduces to

$$(q^2 + H)^3 = 0.$$

Hence $G^2 + 4H^3 = 0$, and the quartic (7) reduces to

$$\{(ax + b + q)^2 - 4q^2\}(ax + b - q)^2 = 0;$$

that is, $$(ax + b - q)^3(ax + b + 3q) = 0,$$

where $$q = +\sqrt{(-H)} \text{ if } G \text{ is positive},$$

and $$q = -\sqrt{(-H)} \text{ if } G \text{ is negative}.$$

When one of the roots of the cubic is zero, that is, when $G = 0$, the equation is reducible at once to a quadratic in $(ax + b)^2$. In fact we have

$$(ax + b)^4 + 6H(ax + b)^2 + a^2I - 3H^2 = 0;$$

therefore $$\{(ax + b)^2 + 3H\}^2 - (12H^2 - a^2I) = 0.$$

Also, in this case,

$$H(12H^2 - a^2I) = 2HI - 3aJ,$$

and $$a^6(I^3 - 27J^2) = (12H^2 - a^2I)^2(a^2I - 3H^2),$$

since $$4H^3 = a^2(HI - aJ).$$

When two of the roots of the cubic are zero (a particular case of equal roots), the quartic becomes a perfect square; for in this case

$$G = 0, \quad 12H^2 - a^2I = 0,$$

and (6) reduces to $$q^4(q^2 + 3H) = 0,$$

whence (7) becomes $$\{(ax + b)^2 + 3H\}^2 = 0.$$

Lastly, when all the roots of the cubic vanish, the biquadratic reduces to

$$(ax + b)^4 = 0;$$

for in this case all the coefficients H, G, F vanish: a condition which may also be written

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}.$$

A METHOD OF EXPRESSING ANY PARTICULAR ARBITRARY CONSTANT IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS IN TERMS OF THE INITIAL CONDITIONS.

Part I.

By E. J. Routh.

THE object of the following paper may be very briefly stated. Given any number of simultaneous differential equations with constant coefficients, it is known that the dependent variables x, y, z, &c. can be expressed in terms of the independent variable t, by means of a series of exponentials real or imaginary. Let one of these exponentials be $x = Pe^{pt}$, then P is a function of the initial values of the variables x, y, &c. and of their differential coefficients. It is here proposed to exhibit this function. Thus, without solving the equations, any one term of the solution can be separated from the others and its value written down, without finding those other terms. The rule is given with some examples in Art. 7, and to this the reader may at once proceed.

When the differential equations are not of a high order we can generally solve the determinantal equation and find all the possible values of p. In such a case it is merely a question of Algebra to find the constants in terms of the initial values of the variables. We may, however, effect this more briefly and simply by using the rule here given. Sometimes it is impossible to solve the determinantal equation. We may find one or more roots, but the rest remain unknown. In such a case we could not proceed by the processes of common Algebra, for the equations cannot be written down. Our object is to find the constants which accompany these known terms without the knowledge of the remaining ones.

This method is very simple and easy of application when the exponential to be separated from the others is connected with a solitary root of the fundamental determinant. But it may be used even though the root is repeated several times. The complication arises from the fact that the exponential is then accompanied by as many constants as there are equal roots. Each of these requires a separate operation to find its value.

The method is generally applicable whatever be the order of the equations, but there is considerable simplification when the order is not higher than the second. This is of course the most interesting case, as the equations may then be such as occur in Dynamics.

In some cases the rule can be put into another form, which may possibly be thought simpler. In these cases it takes the form of the Method of Multipliers. The case in which the fundamental determinant is symmetrical has been chosen as an example. This occurs when we use Lagrange's equations for the oscillations of a system about a position of equilibrium, even when these oscillations are complicated by resistances which vary as the first power of the velocities. When the number of dependent variables is infinite, we have an example in Fourier's rule to expand any function in a series of sines or cosines.

One Dependent Variable.

1. Before considering the general problem in which we have several dependent variables to be found from as many equations, let us take the simpler case in which we have a single equation of the n^{th} order to find x as a function of t.

Let this equation be

$$(A_n\delta^n + A_{n-1}\delta^{n-1} + \ldots + A_1\delta + A_0)x = 0,$$

where the symbol δ represents differentiation with regard to t. We shall write this equation more briefly

$$f(\delta)\, x = 0.$$

Suppose the solution to be

$$x = Pe^{pt} + Qe^{qt} + \ldots,$$

where of course $f(p) = 0, f(q) = 0$, &c.

We notice that

$$\frac{f(\delta) - f(p)}{\delta - p}\, e^{qt} = \frac{f(q) - f(p)}{q - p}\, e^{qt} = 0,$$

except when $q = p$. If $q = p$ the result is evidently $f'(p) e^{pt}$. It immediately follows that

$$Pf'(p) e^{pt} = \frac{f(\delta) - f(p)}{\delta - p}\, x.$$

This is of course equivalent to a first integral of the differential equation, P being the constant introduced by integration.

Putting $t = 0$, and supposing x, δx, &c. to have their known initial values, we find

$$Pf'(p) = \frac{f(\delta) - f(p)}{\delta - p}\, x.$$

Since $f(p) = 0$ and $f(\delta)$ is divisible by $\delta - p$ without remainders, we may write this in the form

$$Pf'(p) = \frac{f(\delta)}{\delta - p}\, x,$$

or writing the operator at length,

$$Pf'(p) = \{A_1 + A_2(\delta + p) + A_3(\delta^2 + \delta p + p^2) + \ldots\}\, x.$$

2. *Example.* Let the given equation be

$$(\delta^3 - 2\delta^2 - \delta + 2)\, x = 0;$$

we notice that this equation is satisfied by $x = Pe^t$. The initial values of x, δx, $\delta^2 x$ being given respectively equal to a, a', a'', it is required to write down the value of the constant P without inquiring into the other solutions.

By the rule we have

$$P(3p^2 - 4p - 1) = \frac{\delta^3 - 2\delta^2 - \delta + 2}{\delta - 1}\, x;$$

therefore

$$\begin{aligned} -2P &= (\delta^2 - \delta - 2)\, x \\ &= a'' - a' - 2a. \end{aligned}$$

If the equation $f(\delta)=0$ have two roots equal to p, both $f(p)=0$ and $f'(p)=0$. Thus our expression for P becomes inapplicable. In the case of two equal roots the solution is known to take the form

$$x=(P_0+P_1t)e^{pt}+Qe^{qt}+\dots .$$

If we operate with $\frac{f(\delta)}{\delta-p}$ as before, and remember that

$$f(\delta)=f(p)+f'(p)(\delta-p)+f''(p)\frac{(\delta-p)^2}{1.2}+\dots,$$

we find
$$\frac{f(\delta)}{\delta-p}x=\frac{f''(p)}{1.2}P_1e^{pt}.$$

If we also operate with $\frac{f(\delta)}{(\delta-p)^2}$ we find

$$\frac{f(\delta)}{(\delta-p)^2}x=\left\{\frac{f''(p)}{1.2}(P_0+P_1t)+\frac{f'''(p)}{1.2.3}P_1\right\}e^{pt}.$$

Putting $t=0$ and supposing x, δx, &c. to have their initial values, we have the following equations to find P_1 and P_2:

$$\frac{f''(p)}{1.2}P_1=\frac{f(\delta)}{\delta-p}x,$$

$$\frac{f''(p)}{1.2}P_0+\frac{f'''(p)}{1.2.3}P_1=\frac{f(\delta)}{(\delta-p)^2}x$$
$$=\frac{d}{dp}\frac{f(\delta)}{\delta-p}x.$$

4. In the same way, if we have three roots each equal to β, we write the solution

$$x=\left(P_0+P_1t+P_2\frac{t^2}{1.2}\right)e^{pt},$$

we then find, if Ln be written for 1.2.3...n,

$$\left.\begin{aligned}
\frac{f'''(p)}{\mathrm{L}3}P_2&=\frac{f(\delta)}{\delta-p}x,\\
\frac{f'''(p)}{\mathrm{L}3}P_1+\frac{f^{iv}(p)}{\mathrm{L}4}P_2&=\frac{d}{dp}\frac{f(\delta)}{\delta-p}x,\\
\frac{f'''(p)}{\mathrm{L}3}P_0+\frac{f^{iv}(p)}{\mathrm{L}4}P_1+\frac{f^{v}(p)}{\mathrm{L}5}P_2&=\frac{1}{\mathrm{L}2}\frac{d^2}{dp^2}\frac{f(\delta)}{\delta-p}x,
\end{aligned}\right\}$$

where we suppose that x, δx, &c. on the right hand side have their initial values.

The same rule may evidently be extended to any number of roots.

5. In many dynamical investigations the equation $f(\delta)x=0$ contains only even powers of δ, so that the roots enter in pairs. Thus we have

$$x = Pe^{pt} + P'e^{-pt} + \&c.$$

If also the motion indicated be oscillatory, p will be imaginary and of the form $p = k\sqrt{(-1)}$. We thus have

$$x = F\cos kt + G\sin kt + \&c.,$$

where $F = P + P'$ and $G = (P - P')\sqrt{(-1)}$.

Now by Art. (1) we have

$$\left.\begin{aligned} Pf'(p) &= \frac{f(\delta)}{\delta - p}x. \\ P'f'(-p) &= \frac{f(\delta)}{\delta + p}x. \end{aligned}\right\}$$

Hence, by addition and subtraction, we find

$$\frac{F}{2}\frac{f'(p)}{p} = \frac{f(\delta)}{\delta^2 - p^2}x,$$

$$\frac{G}{2}\frac{f'(p)}{p} = \frac{f(\delta)}{\delta^2 - p^2}\frac{\delta x}{k}.$$

6. In most dynamical problems the roots of the equation $f(p)=0$ are either negative or have the real parts of the imaginary roots negative. It may also not be convenient to solve the equation $f(p)=0$ so as to find the values of the real roots, and therefore of course we cannot apply the rules given above to find the actual magnitude of the coefficient corresponding to any root. But we may yet find a *superior limit to the magnitude of the coefficient* of any real exponent.

Let $x = Pe^{pt}$ be any real exponential in the solution, then we may write

$$Pf'(p) = (A_1 + A_2 p + A_3 p^2 + \ldots)x + (A_2 + A_3 p + \ldots)\delta x + (A_3 + A_4 p + \ldots)\delta^2 x + \ldots$$

Now when the real parts of the roots of an equation are all negative, the coefficients of that equation must be all positive. But the roots of the equation

$$\frac{f(z)}{z-p} \equiv (A_1 + \&c.) + (A_2 + \&c.)z + \ldots + A_n z^n = 0$$

possess this peculiarity. Hence the coefficients are all positive. But these are the same as the coefficients of x, δx, &c. in the expression for $Pf'(p)$. It immediately follows that any one of these coefficients is less than its first term. For take any coefficient (say the first) and write it in the form

$$A_1 + p(A_2 + A_3 p + \ldots).$$

Here p is negative and the quantity in brackets is positive. Hence the whole expression is less than A_1.

It appears therefore that if we take the expression

$$\frac{A_1 x + A_2 \delta x + \ldots + A_n \delta^{n-1} x}{f'(p)},$$

and omit those terms in the numerator which have an opposite sign to $f'(p)$, we have a superior limit to the value of the coefficient of the exponential e^{pt} in the solution. Here of course x, δx, &c. are all supposed to have their known initial values.

To apply the rule we must have an inferior limit to the value of $f'(p)$. If we know the value of p *approximately* we may find an inferior limit by using the approximate formula

$$f'(p) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha},$$

where α and β are any quantities nearly equal to p.

Any Number of Dependent Variables.

7. Let us now examine the more general case in which we have n equations to find x, y, z, u, v, &c., which we may write

$$\left.\begin{aligned} f_{11}(\delta)x + f_{12}(\delta)y + f_{13}(\delta)z + \ldots &= 0, \\ f_{21}(\delta)x + f_{22}(\delta)y + f_{23}(\delta)z + \ldots &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= 0. \end{aligned}\right\}$$

In dynamical applications these functions of δ are all of the second degree, but at present we make no restriction of that kind.

To solve these we form the determinant

$$\Delta(p)=\begin{vmatrix} f_{11}(p), & f_{12}(p), & f_{13}(p) \\ f_{21}(p), & f_{22}(p), & f_{23}(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix}.$$

If we equate this determinant to zero, we have an equation to find p. Let its roots be p, q, &c. Then we know that

$$x=Pe^{pt}+Qe^{qt}+\ldots.$$

It is our present object to find any one of these coefficients, say P, without finding any of the others. It may be that the equation $\Delta(p)=0$ cannot be completely solved, so that the form of the terms which depend on the roots other than p cannot be found. Our object, therefore, is to find P without using the remaining terms of this series.

To effect this, we deduce from the determinant $\Delta(p)$ another determinant, which we write

$$\Pi(p)=\begin{vmatrix} \dfrac{f_{11}\delta-f_{11}p}{\delta-p}x+\dfrac{f_{12}\delta-f_{12}p}{\delta-p}y+\&c., & f_{12}(p), & f_{13}(p), & \&c. \\ \dfrac{f_{21}\delta-f_{21}p}{\delta-p}x+\dfrac{f_{22}\delta-f_{22}p}{\delta-p}y+\&c., & f_{22}(p), & f_{23}(p), & \&c. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix}.$$

We form this determinant by the following rule: *Erase any column of the determinant $\Delta(p)$, say the first column. To replace it we divide the first equation by $\delta-p$, and, rejecting the remainder, place the quotient in the first row of the erased column. We divide the second equation by $\delta-p$, and place the quotient in the second row, and so on.*

If we erase the second column of the determinant $\Delta(p)$, we obtain a slightly different determinant, which we may write $\Pi_2(p)$, the suffix indicating which column of $\Delta(p)$ we erase.

We may now write the required value of P in the form

$$P\Delta'(p)=\Pi(p).$$

In the same way, if P_2e^{pt} be any term in the value of y, we have

$$P_2\Delta'(p)=\Pi_2(p).$$

Here, of course, $\Delta'(p)$ is the differential coefficient of $\Delta(p)$.

8. Before proceeding to the demonstration of this theorem let us consider an example. Taking the equations

$$\left.\begin{aligned}(\delta^2-4\delta)x-(\delta-1)y&=0\\(\delta+6)x+(\delta^2-\delta)y&=0\end{aligned}\right\},$$

we see that the fundamental determinant

$$\Delta(p) = \begin{vmatrix} p^2 - 4p, & -(p-1) \\ p + 6, & p^2 - p \end{vmatrix}$$
$$= p^4 - 5p^3 + 5p^2 + 5p - 6.$$

Equating this to zero, we find that one value of p is $p = -1$. Let us find the coefficient of e^{-t} in the value of x.

Dividing the equations by $\delta + 1$ and rejecting the remainders, we form at once the second determinant, viz.

$$\Pi(p) = \begin{vmatrix} (\delta - 5)x - y, & 2 \\ x + (\delta - 2)y, & 2 \end{vmatrix},$$

the second column being obtained by putting $p = -1$ in the second column of $\Delta(p)$. Expanding, and noticing that $\Delta'(p) = -24$ when $p = -1$, we find

$$-12P = \delta x - \delta y - 6x + y,$$

where P is the required coefficient. Here x, y, δx, δy are supposed to have their known initial values.

Example 2. Let us take another example, in which the differential coefficients rise to a higher order, but let us still restrict ourselves to two dependent variables to save space. Taking the equations

$$\left.\begin{aligned} (\delta^3 + 2\delta^2 + \delta + 1)x + (\delta^3 + 2\delta + 1)y &= 0 \\ (\delta^2 + 2\delta \quad\quad + 2)x + (\delta^4 + \quad \delta + 2)y &= 0 \end{aligned}\right\},$$

we see by inspection that the determinantal equation is satisfied by $p = 1$. Thus $x = Pe^t$ is a part of the solution. Let it be required to find P, when the initial values of δx, $\delta^2 x$, δy, $\delta^2 y$, $\delta^3 y$ are all zero, and the initial values of x and y unity. Constructing the function Π by dividing each equation by $\delta - 1$, and putting $\delta = 0$ as we proceed, we have

$$\Pi(p) \equiv \begin{vmatrix} 4x + 3y, & 4 \\ 3x + 2y, & 4 \end{vmatrix} = P.\Delta'(p).$$

But, differentiating the determinant without expanding it, and putting $p = 1$, we have $\Delta'(p) = 16$. Hence, putting x and y each equal to unity, we immediately find $P = \frac{1}{2}$.

9. There is another form in which the operator $\Pi(p)$ may be written. If we notice that the determinant $\Delta(p)$

is zero, we see by a simple reduction that

$$\Pi(p) = \frac{1}{\delta - p} \begin{vmatrix} f_{11}(\delta)\, x + f_{12}(\delta)\, y + \&c., & f_{12}(p), & f_{13}(p), & \&c. \\ f_{21}(\delta)\, x + f_{22}(\delta)\, y + \&c., & f_{22}(p), & f_{23}(p), & \&c. \\ \cdots\cdots\cdots\

we obtain

$$e^{qt}G\begin{vmatrix} \frac{f_{11}(q)-f_{11}(p)}{q-p}M_{11}(q)+\frac{f_{12}(q)-f_{12}(p)}{q-p}M_{12}(q)\ \&c.,\ f_{12}(p),\ \&c. \\ \frac{f_{21}(q)-f_{21}(p)}{q-p}M_{11}(q)+\frac{f_{22}(q)-f_{22}(p)}{q-p}M_{12}(q)\ \&c.,\ f_{22}(p),\ \&c. \\ \ldots\ldots\ldots\

ferentiating numerator and denominator in the usual manner, we find that *if p be any root of* $\Delta(p)=0$,

$$\Pi(p)\left[GM_{11}(p)\,e^{pt}\right]=GM_{11}(p)\,e^{pt}\Delta'(p).$$

It follows, therefore, that if we operate with $\Pi(p)$ on any exponential of the solution except e^{pt}, the result is zero. If there be several roots equal to q, these may be regarded as the limiting case of unequal roots, q, $q+h$, &c., where h, &c. is indefinitely small. Hence, whatever the other exponentials may be, we can separate an exponential which depends on a solitary root p by using the operator $\Pi(p)$. If the term to be separated be $x=Pe^{pt}$, the result is

$$\Pi(p)=Pe^{pt}\Delta'(p).$$

Hence, putting $t=0$, and giving x, δx, &c. as they occur in $\Pi(p)$ their initial values, we have

$$P\Delta'(p)=\Pi(p),$$

which is the result to be proved.

13. Before proceeding further it will be convenient to examine another mode of proof, which has the advantage of leading us to the proper form for Π, which appears to be rather arbitrarily introduced in Art. 7.

Let us multiply each of the equations in Art. 7 by e^{-pt}, and then integrate each by parts. Here p is a quantity at our disposal, and is not necessarily a root of the determinant $\Delta(p)$.

The following theorem admits of easy verification, and enables us to effect the integration at once. We have

$$\int e^{-pt}f(\delta)\,x\,dt=\frac{f(\delta)-f(p)}{\delta-p}\,xe^{-pt}+f(p)\int e^{-pt}x\,dt.$$

Applying this theorem, the unintegrated portion from the first equation is

$$\int\{f_{11}(p)\,x+f_{12}(p)\,y+\ldots\}\,e^{-pt}dt,$$

and the integrated portion is

$$\left\{\frac{f_{11}\delta-f_{11}p}{\delta-p}\,x+\frac{f_{12}\delta-f_{12}p}{\delta-p}\,y+\ldots\right\}e^{-pt}.$$

We have similar results from the other equations.

It is clear that if we multiply the unintegrated portions of the several equations by the minors of the constituents in the first column of the determinant $\Delta(p)$ and add the

results, the sum will be

$$\Delta(p)\int x e^{-pt}dt.$$

At the same time the sum of the integrated portions will be $\Pi(p)e^{-pt}$, where $\Pi(p)$ stands for the determinant given in Art. 7. On the right-hand side of the equation we shall have some constant C, which is a function of p, and also of any constants which may enter into the values of x, y, &c.

Multiplying both sides by e^{pt}, we thus obtain the equation

$$\Pi(p)+\Delta(p)e^{pt}\int x e^{-pt}dt = Ce^{pt}.$$

14. If $\Delta(p)=0$, we see from the equation just proved that whatever solution we write for x, y, the operator $\Pi(p)$ will produce zero unless the solution be such as to produce an exponential e^{pt}; also if $\Delta(p)=0$ have m roots equal to p, we see, by differentiation, that the operator $\dfrac{d^\alpha \Pi(p)}{dp^\alpha}$, *where* α *is less than* m, will produce zero or terms of the form

$$(C_0+C_1t+\ldots+C_\alpha t^\alpha)e^{pt}.$$

Hence, whatever form the rest of the values of x, y, &c. may assume, if $\Delta(p)=0$ have m roots equal to p, the constants in

$$x=(P_0+P_1t+\ldots+P_{m-1}t^{m-1})e^{pt},$$

may be found by the use of the operators

$$\Pi(p),\ \frac{d\Pi(p)}{dp},\ \ldots\ \frac{d^{m-1}\Pi(p)}{dp^{m-1}}.$$

By performing these operations we separate the solution depending on the root p from all other solutions.

Taking for example $\Pi(p)$, *its value is known in terms of the initial conditions. But substituting for* x, y, *&c. the parts of their values which depend on the root* p, *we find* $\Pi(p)$ *in terms of the constants connected with that root. This therefore supplies one equation to find these constants.*

15. *Example.* Let the equations be

$$\left.\begin{aligned}(\delta^2-3\delta+2)x+(\delta-1)y&=0\\ -(\delta-1)x+(\delta^2-5\delta+4)y&=0\end{aligned}\right\}.$$

Supposing it to be known that the fundamental determinant has two roots equal to 3, it is required to find the terms in *both* x *and* y which depend on this root.

Let the value of x be taken as

$$x = (P_0 + P_1 t)\, e^{st},$$

then, by substitution, we find

$$y = -(P_0 + P_1 + P_1 t)\, e^{st}.$$

We now form the operator $\Pi(p)$ by Art. 9, copying the columns from the equations given above.

$$\Pi(p) = \begin{vmatrix} \delta x - \dfrac{2x-y}{p}, & p-1 \\ \delta y - \dfrac{x+4y}{p}, & p^2-5p+4 \end{vmatrix}$$

$$= \{(p-4)\,\delta x - \delta y - \frac{2p-9}{p}\,x + y\}\,(p-1).$$

This gives, when $p=3$,

$$\left.\begin{aligned} \Pi(p) &= -2\,(\delta x + \delta y - x - y) \\ \frac{d\Pi(p)}{dp} &= \delta x - \delta y - x + y \end{aligned}\right\}.$$

Substituting for x and y the values written above, we find

$$\left.\begin{aligned} -2\,(\delta x + \delta y - x - y) &= 4P_1 e^{3t} \\ \delta x - \delta y - x + y &= 4\,(P_0 + P_1 + P_1 t)\, e^{3t} \end{aligned}\right\},$$

putting $t=0$, and giving x, y, δx, δy their initial values, we have two equations to find P_0 and P_1.

16. *Application to the method of multipliers.* We shall now make an application of the operator $\Pi(p)$ which will put the method in a new light.

Let us suppose that the given differential equations are such that their fundamental determinant is symmetrical about the leading diagonal. In this case the minors of the constituents in any column are equal to the minors of the constituents in the corresponding row.

Let the solution be written in the form

$$x = P_1 e^{pt} + Q_1 e^{qt} + \ldots,$$
$$y = P_2 e^{pt} + Q_2 e^{qt} + \ldots,$$
$$z = \&c.$$

Then we have

$$\frac{P_1}{M_{11}(p)} = \frac{P_2}{M_{12}(p)} = \&c. = G,$$

$$\frac{Q_1}{M_{11}(q)} = \frac{Q_2}{M_{12}(q)} = \&c. = H,$$

where G, H, &c. are the arbitrary constants.

To shorten the explanation, let us suppose that the equations of motion are of the second order. Let us first choose the second form of $\Pi(p)$ given in Art. 9. In this case we see that if we expand $\Pi(p)$, we obtain

$$G\Pi(p) = (A_{11}\delta x + \&c.)\, P_1 + (A_{12}\delta x + \&c.)\, P_2 + \&c.$$

This may be written

$$= (A_{11}P_1 + A_{12}P_2 + \&c.)\, \delta x + (A_{12}P_1 + A_{22}P_2 + \&c.)\, \delta y + \&c.$$

$$- (C_{11}P_1 + C_{12}P_2 + \&c.)\, \frac{x}{p} - (C_{12}P_1 + C_{22}P_2 + \&c.)\, \frac{y}{p} - \&c.$$

Let us use A and C as functional symbols, thus

$$\left.\begin{aligned} A(P) &= \tfrac{1}{2}A_{11}P_1^2 + A_{12}P_1P_2 + \&c. \\ C(P) &= \tfrac{1}{2}C_{11}P_1^2 + C_{12}P_1P_2 + \&c. \end{aligned}\right\}.$$

Then we have, writing A for $A(P)$, &c.

$$G\Pi(p) = \frac{dA}{dP_1}\delta x + \frac{dA}{dP_2}\delta y + \ldots$$

$$- \frac{dC}{dP_1}\frac{x}{p} - \frac{dC}{dP_2}\frac{y}{p} - \ldots.$$

This result gives a theorem in what we may call the *method of multipliers.*

17. Suppose we have a set of equations of motion such as Lagrange's equations with any number of coordinates, or even an infinite number of coordinates, as in the case of vibrating strings. These equations may contain the *forces of resistance*, i.e. the forces which vary as $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c. The sole restriction here made is, that the fundamental determinant of the motion should be symmetrical.

These equations being solved, we can write the solution in the form of oscillations, or arranged according to exponentials with real or imaginary exponents. The relations between the coefficients which belong to any one exponential are known by the method of solution. But each exponential is accompanied by one unknown constant (called G or H, &c. above). The value of this constant depends on the initial conditions.

To determine this constant we use the method of multipliers. We multiply the expressions x, y, &c.; δx, δy, &c. by certain multipliers, and add the results together. These

multipliers are to be such that all the exponentials, except the one we wish to retain, shall disappear from the sum. Then the initial values of δx &c., x &c. being known, the value of the constant becomes apparent.

The difficulty is to find these multipliers. By Fourier's theorem they are known for such cases as strings, &c., when the *vis viva* takes the form of the sum of the squares of the differential coefficients of the coordinates. But this value of $\Pi(p)$ gives these multipliers in every case in which the fundamental determinant is symmetrical.

The proper multiplier for δx is $\frac{dA}{dP_1}$; for δy, $\frac{dA}{dP_2}$; and so on, as indicated in the expression for $\Pi(p)$. Applying this rule, we find

$$\frac{dA}{dP_1}\delta x + \&c. - \frac{dC}{dP_1}\frac{x}{p} - \&c. = 2\left(pA - \frac{C}{p}\right)e^{pt}.$$

Putting $t=0$ and giving x, δx, &c. their initial values, we have an equation to find G.

We may remark that by Art. 7 we must have the coefficient of the exponential on the right-hand side

$$= G^2 M_{11}(p)\,\Delta'(p),$$

a result which may be easily verified.

18. *To use the method in this form with convenience we must have a rule to write down the functions A and C, whence the multipliers can be found by differentiation.*

To find A and C multiply the given equations by x, y, z, &c., and add the results. Half the coefficient of δ^2 is the function A with x, y, &c. replacing P_1, P_2, &c. Half the term independent of δ is the function C with x, y, &c. replacing P_1, P_2, &c. The proper multipliers for δx, δy, &c.; x, y, &c., are $\frac{dA}{dx}$, $\frac{dA}{dx}$, &c.; $-\frac{dC}{pdx}$, $-\frac{dC}{pdy}$, &c., when we have written P_1, P_2, ... for x, y, &c.

19. *Example.* The differential equations

$$\left.\begin{aligned}(\delta^2 - 2\delta + 2)\,x + \sqrt{5}\,\delta y = 0\\ \sqrt{5}\,\delta x + (\delta^2 + 2\delta + 2)\,y = 0\end{aligned}\right\}$$

admit of a solution of the form

$$x = P_1 e^t, \quad y = -\frac{P_1}{\sqrt{5}}e^t = P_2 e^t.$$

The initial values of x, y, δx, δy being known, find the constant P_1 without finding the rest of the solution.

We first find the functions $A(x)$ and $C(x)$ by multiplying the equations by x and y, and adding the results (Art. 18). The terms with δ^2 give at once

$$2A(x) = x^2 + y^2.$$

The terms without δ give

$$2C(x) = 2x^2 + 2y^2.$$

The proper multipliers for δx, δy, x, and y respectively are (Art. 17)

$$\frac{dA}{dP_1} = P_1, \quad \frac{dA}{dP_2} = P_2, \quad -\frac{dC}{pdP_1} = -2P_1, \quad -\frac{dC}{pdP_2} = -2P_2.$$

Hence we find

$$P_1\delta x + P_2\delta y - 2P_1 x - 2P_2 y = -\tfrac{6}{5}P_1^2 e^t.$$

Substituting for P_2 in terms of P_1, we find

$$\delta x - \frac{1}{\sqrt{5}}\delta y - 2x + \frac{2}{\sqrt{5}}y = -\tfrac{6}{5}P_1 e^t.$$

If we now put $t=0$, and give x, δx, &c. their initial values, the constant P_1 has been found.

20. When the forces of resistance are absent we have the ordinary case of Lagrange's equations. The function A is then analogous to the kinetic energy and C to the potential energy. In this case, since the terms $B_{11}\delta$, $B_{12}\delta$, &c. are absent from the equations, it is more convenient to deduce the function $\Pi(p)$ from the form in Art. 7. We then find

$$G\Pi(p) = \frac{dA}{dP_1}\delta x + \frac{dA}{dP_2}\delta y + \dots .$$

If we divide each equation by δ^2 before forming the operator $\Pi(p)$ as explained in Art. 9, we find

$$-Gp^2\Pi(p) = \frac{dC}{dP_1}x + \frac{dC}{dP_2}\delta y + \dots .$$

Thus in Lagrange's equations *we have a choice of two sets of multipliers*, either of which will enable us to separate any oscillation from all the others. We may use either the multipliers

$$\frac{dA}{dP_1}, \quad \frac{dA}{dP_2}, \quad \&c.,$$

obtained from the kinetic energy, or the multipliers

$$\frac{dC}{dP_1}, \quad \frac{dC}{dP_2}, \quad \&c.,$$

obtained from the potential energy.

21. If either the kinetic energy or the potential energy can be written in the form

$$\tfrac{1}{2}(\delta x)^2 + \tfrac{1}{2}(\delta y)^2 + \ldots \quad \text{or} \quad \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \ldots,$$

we fall back on the old rule usually called Fourier's rule, which may be briefly expressed thus:—To separate the exponential e^{pt} from the others, we multiply the values of x, y, z, &c. by the coefficient of the exponential in that equation and add the results. Thus we find

$$P_1 x + P_2 y + \ldots = (P_1^2 + P_2^2 + \ldots)\, e^{pt}.$$

22. When the determinant of the motion is not symmetrical about the leading diagonal, the function $\Pi(p)$ will still enable us to find the proper multipliers whenever such multipliers exist. But this we pass over for the moment and reserve for another communication. We shall next proceed to complete the theory of the operator $\Pi(p)$ by considering its application to the case of equal roots.

Part II.

TREATMENT OF EQUAL ROOTS.

Minors not Zero.

23. When equal roots occur in the determinant $\Delta(p)$, the rule to find any *single coefficient* in terms of the initial conditions is not so simple. Two cases occur which may be considered separately.

Case 1. These equal roots are *not supposed to make all the first minors of the determinant* $\Delta(p)$ *equal to zero.* In this case the values of x, y, &c. will contain terms of the form $t^n e^{pt}$ as wells as terms of the form e^{pt}.

Case 2. These equal roots *are supposed to make all the first minors equal to zero.* Perhaps all the second or third minors may also vanish. In this case the terms containing

the higher powers of t which should otherwise enter into the values of x, y, &c. do not appear. Their place is occupied by an increased number of unknown constants accompanying the remaining exponentials.

24. Taking these cases in order, let us suppose that the determinantal equation $\Delta(p) = 0$ has m roots p, q, r, &c., which we shall presently take equal to each other. The part of the solution which depends on these roots is

$$x = GM_{11}(p)\, e^{pt} + HM_{11}(q)\, e^{qt} + KM_{11}(r)\, e^{rt} + \dots,$$
$$y = GM_{12}(p)\, e^{pt} + HM_{12}(q)\, e^{qt} + KM_{12}(r)\, e^{rt} + \dots,$$
$$z = \&c.$$

where G, H, &c. are the arbitrary constants, and the functional symbol M represents a minor of Δ.

If we now put $q = p + h$, $r = p + k$, &c., and take the limit when h, k, &c. are zero, these solutions take the following forms analogous to those usually given for one dependent variable.

$$x = Q_0 M_{11}(p)\, e^{pt} + Q_1 \frac{d}{dp}[M_{11}(p)\, e^{pt}] + Q_2 \frac{d^2}{dp^2}[M_{11}(p)\, e^{pt}] + \dots,$$

$$y = Q_0 M_{12}(p)\, e^{pt} + Q \frac{d}{dp}[M_{12}(p)\, e^{pt}] + Q_2 \frac{d^2}{dp^2}[M_{12}(p)\, e^{pt}] + \dots,$$

$$z = \&c.$$

where $Q_0 = G + H + \&c.$, $\quad Q_1 = Gh + Hk + \&c.$, $\quad Q_2 = \&c.$,

so that Q_0, $Q_1 \dots Q_{m-1}$ are the m arbitrary constants required to give a complete solution.

25. We have now to investigate a method of finding these constants Q_0, $Q_1 \dots Q_{m-1}$ when the initial values of x, y, &c., δx, &c. are known.

To effect this we use the equation proved in Art. 10,

$$\Pi(p)\,[M_{11}(q)\, e^{qt}] = e^{qt}\, \frac{\Delta(q)\, M_{11}(p) - \Delta(p)\, M_{11}(q)}{q - p},$$

where p and q are unrestricted in value. Differentiating this with regard to p and q by Leibnitz's theorem, we see that

$$\frac{d^\alpha \Pi(p)}{dp^\alpha} \left[\frac{d^\beta}{dq^\beta} M_{11}(q)\, e^{qt}\right] = 0,$$

provided p and q are *unequal*, and

$$\Delta(p)=0, \quad \Delta'(p)=0, \text{ \&c.},$$

so that there are at least $\alpha+1$ roots equal to p, and also

$$\Delta(q)=0, \quad \Delta'(q)=0, \text{ \&c.},$$

so that there are at least $\beta+1$ roots equal to q.

Suppose then that the determinantal equation $\Delta(p)=0$ has m roots equal to p. Then we see that if we perform on x, y, &c., the operations indicated by

$$\Pi(p), \quad \frac{d\Pi(p)}{dp}, \quad \frac{d^2\Pi(p)}{dp^2} \cdots \frac{d^{m-1}\Pi(p)}{dp^{m-1}},$$

then all the terms in x, y, &c. which depend on the roots of the determinantal equation other than p will disappear from the results. These results will contain only the m constants $Q_0, Q_1 \cdots Q_{m-1}$. [See Art. 14 for another proof.]

In practice we may find these results in each separate problem by actually substituting these particular terms of x, y in the operators. Or we may use a rule which will be presently given to write down the result at once.

Whichever method we adopt, we have sufficient equations to find $Q_0 \cdots Q_{m-1}$.

26. If we wish to find the coefficients of the terms in the expression for x, *without knowing the corresponding terms in the expressions for* y, z, *&c.*, we may use a rule which we now proceed to investigate.

Referring to Art. 25, we see that we must find the value when $p=q$ of

$$\frac{1}{\mathrm{L}\alpha\ \mathrm{L}\beta} \frac{d^\alpha \Pi(p)}{dp^\alpha} \left[\frac{d^\beta}{dq^\beta} M_{11}(q)\, e^{qt}\right].$$

Referring to Art. 10, we notice, by Taylor's theorem, that this is the coefficient of $h^\alpha k^\beta$ in the expansion of

$$\frac{\Delta(q+k)\, M_{11}(p+h) - \Delta(p+h)\, M_{11}(q+k)}{q+k-p-h}\, e^{(q+k)t}.$$

We shall neglect the exponential, as only the initial value is required, and omit the suffix of M_{11} for brevity.

Since we have $q=p$, this is the same as the coefficient of $h^\alpha k^\beta$ in the expansion of

$$\frac{\Delta(p+k)\, M(p+h) - \Delta(p+h)\, M(p+k)}{k-h}.$$

We shall represent this required coefficient by X.

Let $Ch^m k^n$ be any term in the expansion of

$$\Delta(p+h)\,M(p+k).$$

Then, interchanging h and k, $Ck^m h^n$ occurs in the expansion of

$$\Delta(p+k)\,M(p+h).$$

Taking these two terms, we obtain

$$C\,\frac{k^m h^n - h^m k^n}{k-h} \doteq C(h^n k^{m-1} + h^{n+1} k^{m-2} + \ldots + h^{m-1} k^n).$$

If then any of the terms in the brackets coincide with $h^\alpha k^\beta$, C will be the required coefficient.

It follows that X is the sum of the coefficients of $h^{\alpha+\beta+1}$, $h^{\alpha+\beta}k$, $h^{\alpha+\beta-1}k^2$, &c., in the expansion of $\Delta(p+h)\,M(p+k)$. That is, putting $h=k$, X is the coefficient of $h^{\alpha+\beta+1}$ in the expansion of $\Delta(p+h)\,M(p+h)$.

The required result is therefore

$$X = M\,\frac{\Delta^{\alpha+\beta+1}}{\mathrm{L}(\alpha+\beta+1)} + \frac{dM}{dp}\,\frac{\Delta^{\alpha+\beta}}{\mathrm{L}(\alpha+\beta)} + \frac{1}{1.2}\,\frac{d^2M}{dp^2}\,\frac{\Delta^{\alpha+\beta-1}}{\mathrm{L}(\alpha+\beta-1)} + \ldots,$$

where the indices of Δ represent differentiation with regard to p, and $\mathrm{L}(n) = 1.2.3\ldots n$.

28. Let us now apply the Lemma just proved to find the value of $\frac{1}{\mathrm{L}\alpha}\,\frac{d^\alpha \Pi(p)}{dp^\alpha}$, when we substitute for x, y, &c. the expressions given in Art. 24, viz.

$$x = Q_0 M_{11}(p)\,e^{pt} + \ldots + Q_{m-1}\,\frac{d^{m-1}}{dp^{m-1}}\,M_{11}(p)\,e^{pt},$$

with similar expressions for y, z, &c. Omitting the suffix for brevity, we find

$$\frac{1}{\mathrm{L}\alpha}\,\frac{d^\alpha \Pi}{dp^\alpha} = Q_{m-1}\,\mathrm{L}(m-1)\left\{M\,\frac{\Delta^{\alpha+m}}{\mathrm{L}(\alpha+m)} + \frac{dM}{dp}\,\frac{\Delta^{\alpha+m-1}}{\mathrm{L}(\alpha+m-1)} + \ldots\right\}$$

$$+ Q_{m-2}\,\mathrm{L}(m-2)\left\{M\,\frac{\Delta^{\alpha+m-1}}{\mathrm{L}(\alpha+m-1)} + \ldots\right\}.$$

But if we expand the differential coefficients in the value of x by Leibnitz's theorem, and obtain

$$x = \left\{P_0 + P_1 t + \ldots + P_{m-1}\,\frac{t^{m-1}}{\mathrm{L}(m-1)}\right\} e^{pt},$$

we see that

$$\frac{P_{m-1}}{\mathrm{L}(m-1)} = MQ_{m-1},$$

$$\frac{P_{m-2}}{\mathrm{L}(m-2)} = MQ_{m-2} + (m-1)\frac{dM}{dp}Q_{m-1},$$

$$\frac{P_{m-3}}{\mathrm{L}(m-3)} = MQ_{m-3} + (m-2)\frac{dM}{dp}Q_{m-2} + \frac{(m-1)(m-2)}{1.2}\frac{d^2M}{dp^2}Q_{m-2}.$$

Hence it immediately follows that

$$\frac{1}{\mathrm{L}\alpha}\frac{d^\alpha \Pi}{dp^\alpha} = \frac{\Delta^{\alpha+m}}{\mathrm{L}(\alpha+m)}P_{m-1} + \frac{\Delta^{\alpha+m-1}}{\mathrm{L}(\alpha+m-1)}P_{m-2} + \dots,$$

where the indices of Δ represent differentiation with regard to p, and $\mathrm{L}\alpha = 1.2.3\dots\alpha$.

29. To illustrate this general formula let us consider the case in which the fundamental determinant $\Delta(p)$ has three roots equal to p. Let the resulting value of x be

$$x = \left\{P_0 + P_1 t + P_2 \frac{t^2}{2}\right\} e^{pt}.$$

Then $\Delta(p) = 0$, $\Delta'(p) = 0$, $\Delta''(p) = 0$; and the values of P_0, P_1, P_2 in terms of the initial values of x, y, &c. are given by

$$\left.\begin{aligned} \frac{\Delta'''}{\mathrm{L}3}P_2 &= \Pi(p), \\ \frac{\Delta'''}{\mathrm{L}3}P_1 + \frac{\Delta^{iv}}{\mathrm{L}4}P_2 &= \frac{d\Pi(p)}{dp}, \\ \frac{\Delta'''}{\mathrm{L}3}P_0 + \frac{\Delta^{iv}}{\mathrm{L}4}P_1 + \frac{\Delta^{v}}{\mathrm{L}5}P_2 &= \frac{1}{1.2}\frac{d^2\Pi(p)}{dp^2}, \end{aligned}\right\}$$

30. As a verification, take the example already discussed in Art. 15. We have

$$\Delta(p) = \begin{vmatrix} p^2 - 3p + 2, & p - 1 \\ -p - 1, & p^2 - 5p + 4 \end{vmatrix} = (p-1)^2(p-3)^2.$$

Hence when $p = 3$, $\Delta(p) = 0$, $\Delta'(p) = 0$, $\Delta''(p) = 8$, and $\Delta'''(p) = 24$. Let the value of x be

$$x = (P_0 + P_1 t)\, e^{3t},$$

that of y not being required. Then, by the rule given in the last article,

$$\left.\begin{aligned} \tfrac{1}{2}\Delta'' P_1 &= -2\,(\delta x + \delta y - x - y) \\ \tfrac{1}{2}\Delta'' P_0 + \tfrac{1}{6}\Delta''' P_1 &= \quad \delta x - \delta y - x + y \end{aligned}\right\},$$

which are the same results as before.

Minors zero.

31. We shall now examine the second case in which the determinantal equation $\Delta(p) = 0$ has equal roots, which at the same time make all the first minors zero. This case occurs frequently in Dynamics, and therefore possesses considerable interest. Taking the case of two equal roots, we see by Art. 24 that, since $M_{11}(p) = 0$, all the terms with t as a factor are absent. The solution which remains, viz.

$$x = Q_1 \frac{dM_{11}(p)}{dp} e^{pt}$$

lacks generality, as we have only one constant, the other Q_0 having disappeared.

To complete the solution it will be convenient to have recourse to the second minors of the fundamental determinant. This forms a contrast between the two cases mentioned in Art. 23. In Case 1 it is convenient to complete our solution by differentiating the first minors; in Case 2 we complete our solution by the aid of the second minors.

32. We shall first show that this second case of equal roots may be treated like the first case as a limit of the general solution with unequal roots, though it may not be convenient always to do so.

To simplify matters, we shall suppose that the equation has two roots each equal to a, which make all the first minors zero. Let us alter the equations by adding on two small arbitrary quantities αx and βy to the constituents of any equation, say the second. Let $p = a + h$, $q = a + k$ be the roots of the determinant $\Delta(p)$ thus modified. Let the solution of the equations be

$$\begin{aligned} x &= GM_{11}(p)\, e^{pt} + HM_{11}(q)\, e^{qt} + \ldots, \\ y &= GM_{12}(p)\, e^{pt} + HM_{12}(q)\, e^{qt} + \ldots, \\ z &= \&c. \end{aligned}$$

Taking any minor $M(p)$, we see that

$$M(p) = M(a) + \alpha \frac{dM(a)}{da} + \beta \frac{dM(\alpha)}{d\beta}.$$

Now M_{11} is not altered by the introduction of α, nor M_{12} by β, while M_{13} is altered by both. Hence, neglecting the squares of small quantities in the limit, we find

$$x = P \frac{dM_{11}(a)}{d\beta} e^{at},$$

$$y = Q \frac{dM_{12}(a)}{da} e^{at},$$

$$z = \left\{ P \frac{dM_{13}(a)}{d\beta} + Q \frac{dM_{13}(a)}{da} \right\} e^{at},$$

where $P = (G + H)\beta$, $Q = (G + H)\alpha$, and therefore are two arbitrary constants.

These differential coefficients of the minors with respect to α or β are really the second minors formed by omitting the row and column in which α or β occurs. But the results written in the form of second minors may be found more conveniently by the following course of reasoning.

33. Let us suppose that the fundamental determinant has m roots equal to p, and that all the first minors have n of these roots, but not all of them have $n + 1$ of these roots. We shall suppose in the first instance that all the second minors do not vanish. Then it may be shown that m is greater than $2n$. *We must first write down the form of the solution depending on these roots in order to discover what the constants are whose values have to be found in terms of the initial conditions.*

Resuming the equations given in Art. 7, let us reject for a moment any two, say the two first equations. Solving the remaining equations to find any variable, say z in terms of x and y, we have

$$|f_{33}(\delta), f_{34}(\delta)\ldots|\, z = -|f_{31}(\delta), f_{34}(\delta)\ldots|\, x - |f_{32}(\delta), f_{34}(\delta)\ldots|\, y.$$

The second minors which form the coefficients of this equation, may be written as differential coefficients of the first minors in the form

$$\frac{dM_{11}(\delta)}{df_{22}} z = \frac{dM_{13}(\delta)}{df_{22}} x - \frac{dM_{13}(\delta)}{df_{21}} y.$$

We notice, in passing, that since

$$\frac{dM_{1r}(\delta)}{df_{2s}} = -\frac{dM_{1s}(\delta)}{df_{2r}},$$

these coefficients may be written in several different forms.

Let now ξ and η be two functions of t, such that if $M(\delta)$ be *any* first minor of the fundamental determinant

$$M(\delta)\,\xi = 0, \quad M(\delta)\,\eta = 0;$$

therefore
$$\left.\begin{aligned}\xi &= (G_0 + G_1 t + \ldots + G_{n-1}t^{n-1})\,e^{pt}\\ \eta &= (H_0 + H_1 t + \ldots + H_{n-1}t^{n-1})\,e^{pt}\end{aligned}\right\}.$$

If then we choose

$$x = \frac{dM_{11}(\delta)}{df_{22}}\,\xi,$$

$$y = -\frac{dM_{11}(\delta)}{df_{22}}\,\eta = \frac{dM_{12}(\delta)}{df_{21}}\,\eta,$$

it follows that

$$z = \frac{dM_{13}(\delta)}{df_{22}}\,\xi + \frac{dM_{13}(\delta)}{df_{21}}\,\eta,$$

and so on through all the variables.

We have yet to consider the two equations rejected in this solution. But since ξ and η make all the first minors zero, these equations are not independent of the rest. It therefore follows that these values of x, y, &c. satisfy all the differential equations.

34. The solution just found has $2n$ constants, viz. the n G's and the n H's. But since m is greater than $2n$, the solution is still incomplete.

This remaining part of the solution depends on the remaining $m-2n$ roots, and will come from that method of solution which does not require the vanishing of the minors. But this has been already found in Art. 24. We have

$$x = Q_0 M_{11}(p)\,e^{pt} + \ldots + Q_{m-1}\frac{d^{m-1}}{dp^{m-1}}M_{11}(p)\,e^{pt},$$

with similar expressions for y, z, &c.

Since the first minors have n roots equal to p, all the terms in this series from Q_0 to Q_{n-1} are zero.

Consider next the solution supplied by any Q_k between Q_n and Q_{2n}. It is an expression of the form

$$(A_0 + A_1 t + \ldots + A_{k-n-1}t^{k-n-1})\,e^{pt}.$$

These may evidently be included in the terms which are derived from ξ. The corresponding terms in y may be included in the η. Thus these terms may be entirely removed from the expressions for x and y. Since the corresponding terms in z (as defined by any coefficient Q_k) must satisfy the equation proved above in (Art. 33), to find z in terms of x and y, it is clear that these terms also are therefore removed from the expression for z. In the same way the corresponding terms are removed from the other variables.

Lastly, consider the solution supplied by the terms from Q_{2n} to Q_{m-1} both inclusive. These terms satisfy the differential equations, and are not included in the solution found above. They also contain the remaining $m-2n$ constants.

The complete solution is therefore found by adding to the terms derived from ξ and η the following

$$x = Q_{2n}\frac{d^{2n}}{dp^{2n}}M_{11}(p)e^{pt} + \ldots + Q_{m-1}\frac{d^{m-1}}{dp^{m-1}}M_{11}(p)e^{pt},$$

with similar expression for y and z, as in Art. 24.

We notice that the terms which are derived from ξ and η supply terms which contain powers of t up to t^{n-1} inclusive. The terms which are derived from the differentials of the minor with regard to p, contain powers of t up to t^{m-n-1}. The powers of t from t^{m-n} to t^{m-1} are absent.

For the sake of brevity, we shall distinguish the terms which are derived from ξ and η as those which contain second minors, while those which contain differentials with regard to p, will be referred to containing first minors.

35. *We have secondly to determine how the operator* $\Pi(p)$ *must be modified that we may be able to separate the terms which depend on these m equal roots from the rest.*

We have already seen that we may use the operators

$$\Pi,\quad \frac{d\Pi}{dp},\ldots\quad \frac{d^{n-1}\Pi}{dp^{n-1}}.$$

Since the differential coefficients of the minors up to the $(n-1)^{\text{th}}$ all vanish, the first n of these operators are absolutely zero. The next $m-2n$ operators will determine by the use of the rule in Art. 29, the coefficients of the powers of t from t^n to t^{m-n-1}, since these powers do not occur in the terms which are derived from ξ and η.

The remaining n operators will act both on the terms which are derived from the second minors, and on those

derived from the first minors. They will give n additional equations to determine the constants. We therefore require at least n additional operators.

36. These additional operators may be found from the equation

$$\Pi(p) + \Delta(p)\, e^{pt} \int e^{-pt} x dt = Ce^{pt},$$

proved in Art. 13. Since we may alter our equations by adding any small arbitrary quantities, such as αx or βy to any equation [Art. 32], it is evident that we may differentiate the equation just written down with regard to any one of the symbols f_{21}, f_{22}, &c. Hence, when $\Delta(p) = 0$, we have

$$\frac{d\Pi}{df} + Me^{pt} \int e^{pt} x dt = \frac{d}{df} Ce^{pt},$$

where f stands for any one of the symbols f_{21} &c., and M is its minor in the determinant $\Delta(p)$.

Since every minor has n roots equal to p, the differential coefficients of M up to the $(n-1)^{\text{th}}$ are zero. Hence we have

$$\frac{d^k}{dp^k} \frac{d\Pi(p)}{df} = \frac{d^k}{dp^k} \frac{d}{df} Ce^{pt},$$

where k is less than n.

It follows from this equation that

$$\frac{d\Pi}{df}, \quad \frac{d}{dp}\frac{d\Pi}{df}, \quad \cdots \frac{d^{n-1}}{dp^{n-1}} \frac{d\Pi}{df}$$

are the operators sought for. They will produce zero when performed on any term except those which depend on the root p. They may therefore be used in the manner explained in Art. 14 to determine the constants which depend on this root.

37. *Example.* Given the initial conditions, find the values of the constants which enter into that part of the solution of the following equations which contains e^t as a factor:

$$\left.\begin{aligned} (\delta^2 - 3\delta + 2)\, x - (\delta - 1)\, y - (\delta - 1)\, z &= 0 \\ -x + (\delta - 2)\, y - \qquad z &= 0 \\ -x - \qquad y + (\delta - 2)\, z &= 0 \end{aligned}\right\}.$$

We see that the fundamental determinant is

$$\Delta p = \begin{vmatrix} (p-1)(p-2), & -(p-1), & -(p-1) \\ -1 & , \quad p-2 & , \quad -1 \\ -1 & , \quad -1 & , \quad p-2 \end{vmatrix} = (p-1)^3(p-4).$$

This has three roots equal to unity, and each minor has one of these.

Then, by Arts. 33 and 34,

$$x = Ge^t + Q\frac{d^2}{dp^2}M_{11}e^{pt} + \ldots$$

$$= Ge^t + 2Q(1-2t)e^t,$$

$$y = He^t + Q\frac{d^2}{dp^2}M_{12}e^{pt} + \ldots$$

$$= He^t + 2Qte^t;$$

thence z follows from the equations

$$z = -(G+H)e^t + 2Qte^t + \ldots.$$

To find the three constants G, H, and Q we shall use the three operators $\frac{d\Pi}{dp}$, $\frac{d^2\Pi}{dp^2}$, and $\frac{d\Pi}{df_{12}}$.

We have, as in Art. 7,

$$\Pi(p) = \begin{vmatrix} (\delta+p-3)x-y-z, & -(p-1), & -(p-1) \\ y & , \quad p-2 & , \quad -1 \\ z, & -1 & , \quad p-2 \end{vmatrix}$$

$$= (p-1)\{(p-3)(\delta+p-3)x + 2y + 2z\}.$$

Thence we find, when $p=1$,

$$\frac{d\Pi}{dp} = -2\{(\delta-2)x - y - z\}.$$

Substituting the values of x, y, z, written down above, this becomes, when t is put zero, $6.2Q$. Thence we have

$$12Q = -2(\delta x - 2x - y - z),$$

where δx, x, y, z have their initial values.

Taking the next operator, we have, when $p=1$,

$$\tfrac{1}{2}\frac{d^2\Pi}{dp^2} = (\delta-4)x.$$

Substituting for x and putting $t=0$, we find

$$-3G-10Q=\delta x-4x.$$

Taking the last operator, we have

$$\frac{d\Pi}{df_{12}}=\begin{vmatrix} y, & -1 \\ z, & p-2 \end{vmatrix}=\begin{vmatrix} y, & -1 \\ z, & -1 \end{vmatrix}.$$

Substituting we find

$$-G-2H=z-y.$$

Thus we have three equations from which we may readily find the three constants G, H, and Q. It will be noticed that they have thus been found without making any use of the parts of x, y, z, which depend on the other root of the equation $\Delta(p)=0$.

38. *If the values of all the variables x, y, z, &c. be not required to be found in terms of t, we must have some method of finding the constants in those required without writing down the others.*

Let us suppose that the constants which occur in x and y, and are in the terms which arise from the m roots equal to p, are required alone.

Let us consider the terms which depend on the second minors. Referring to Art. 33, we notice that all the terms which are derived from ξ may be found just as if $\eta=0$, provided we replaced the fundamental determinant $\Delta(p)$ by the minor of f_{22}. To find these terms, we therefore use the operator $\Pi(p)$ with the column and row containing f_{22} removed. We may retain the terms in the first column which operate on y, because y is zero when we operate on ξ alone. In the same way when we operate on η alone we have a similar operator with the row and column containing f_{21} removed. The two operators may be written as one, viz.

$$\begin{vmatrix} \dfrac{f_{11}\delta-f_{11}p}{\delta-p}x+\dfrac{f_{12}\delta-f_{12}p}{\delta-p}y+\&c., & f_{13}p, & f_{14}p, & \&c. \\ \dfrac{f_{31}\delta-f_{31}p}{\delta-p}x+\dfrac{f_{32}\delta-f_{32}p}{\delta-p}y+\&c., & f_{33}p, & f_{34}p, & \&c. \\ \cdots\cdots\cdots\

For example, suppose that the first minors have *only one* root equal to p. Then taking ξ only, we have

$$x = G_0 \frac{dM_{11}\delta}{df_{22}} e^{pt}, \qquad y = 0, \qquad z = G_0 \frac{dM_{13}\delta}{df_{22}} e^{pt},$$

which we abbreviate into

$$x = Ke^{pt}, \qquad y = 0, \qquad z = K'e^{pt}.$$

Then by the result given in Art. 7,

$$K \cdot \frac{d}{dp} \frac{d\Delta}{df_{22}} = \frac{d\Pi}{df_{22}}.$$

Next, taking η only, we have

$$x = 0, \qquad y = H_0 \frac{dM_{12}\delta}{df_{21}} e^{pt}, \qquad z = H \frac{dM_{13}\delta}{df_{21}} e^{pt}.$$

which we abbreviate into

$$x = Le^{pt} \qquad y = 0, \qquad z = L'e^{pt};$$

and again, by Art. 7,

$$L \frac{d}{dp} \frac{d\Delta}{df_{21}} = \frac{d\Pi}{df_{21}}.$$

When we operate on both the quantities ξ and η, we have

$$K \frac{d}{dp} \frac{d\Delta}{df_{22}} - L \frac{d}{dp} \frac{d\Delta}{df_{21}} = \frac{d\Pi(p)}{df_{22}}.$$

In the same way by removing some other row, say the third from the determinants, we obtain another equation

$$K \frac{d}{dp} \frac{d\Delta}{df_{32}} - L \frac{d}{dp} \frac{d\Delta}{df_{31}} = \frac{d\Pi(p)}{df_{32}}.$$

Taking the general case in which the first minors have n roots equal to p, the terms in Art. 33, which depend on the second minors, may be written

$$x = (P_0 + P_1 t + \ldots + P_{n-1} t^{n-1}) e^{pt},$$
$$y = (Q_0 + Q_1 t + \ldots + Q_{n-1} t^{n-1}) e^{pt}.$$

Then by the rule in Art. 29,

$$\frac{P_{n-1}}{\mathrm{L}n} \frac{d^n}{dp^n} \frac{d\Delta p}{df_{22}} - \frac{Q_{n-1}}{\mathrm{L}n} \frac{d^n}{dp^n} \frac{d\Delta p}{df_{21}} = \frac{d\Pi(p)}{df_{22}},$$

and so on through all the other equations.

39. The terms which arise from the m roots equal to p have been separated in Art. (34) into two classes, (1) those which involve the second minors, and (2) those which depend on first minors alone. The effect of the operators $\frac{d\Pi}{df_{22}}$ and $\frac{d\Pi}{df_{21}}$ and their differentials with regard to p on the terms of the first class have just been found. We must now find their effect on those of the second class, and finally add the results together.

To effect this we have recourse to the identity established in Art. 10,

$$\Pi(p)\left[M(q)\,e^{qt}\right] = e^{qt}\,\frac{\Delta(q)\,M_{11}(p) - \Delta(p)\,M_{11}(q)}{q-p},$$

where p and q are any quantities not necessarily roots of the equation $\Delta p = 0$. Let us now add on to any constituent f of $\Delta(p)$, a small quantity df whose square may be neglected. In this way both the operator Π and the subject M are altered. Subtracting the original equation, we have

$$\frac{d\Pi(p)}{df}\left[Mq\,e^{qt}\right] + \Pi(p)\left[\frac{dM}{df}\,e^{qt}\right] = \frac{e^{qt}}{q-p}\left\{\begin{array}{c} \Delta q\,\dfrac{dMp}{df} + \dfrac{d\Delta q}{df}\,Mp \\ -\Delta p\,\dfrac{dMq}{df} - \dfrac{d\Delta p}{df}\,Mq \end{array}\right\}.$$

We shall now take the α^{th} differential coefficient with regard to p and β^{th} differential with regard to q, and then make $p = q$ and regard each as a root of $\Delta p = 0$, which is m times repeated. Since the minors of Δp contain this root n times, the operations indicated by Π and its differential coefficients up to the $(m-1)^{\text{th}}$ are absolutely zero whatever be the subject of operation. Again, considering the right-hand side, we may obtain its value by the same reasoning as in Art. 27. Hence, we find,

$$\frac{1}{\mathrm{L}\alpha\,\mathrm{L}\beta}\,\frac{d^{\alpha}}{dp^{\alpha}}\,\frac{d\Pi(p)}{df}\left[\frac{d^{\beta}}{dp^{\beta}}\,M_{11}e^{pt}\right]$$

is equal to the coefficient of $h^{\alpha+\beta+1}$ in the expansion of

$$\Delta(p+h)\,\frac{d}{df}M_{11}(p+h) + M_{11}(p+h)\,\frac{d}{df}\Delta(p+h).$$

This may be easily expressed, as in Art. 27, in terms of the differential coefficients of Δp and $M(p)$, but the result is rather long. The lowest power of h which occurs in $\Delta(p+h)$ is h^m, and the lowest power in the second term is h^{2n}. Unless therefore $\alpha+\beta+1$ is greater than $2n$, this result is zero. We may therefore in general calculate the required coefficient more simply from this form than when expressed in a series of differential coefficients of $\Delta(p)$ and $M(p)$.

The rule to predict the result of performing the operator when we have a complicated case of equal roots has been made to depend on the two results given in Arts. 38 and 39, which have to be added together. As it will generally be simpler to follow the method indicated in Art. 36 and illustrated in the example of Art. 37, it seems unnecessary to examine this result more minutely.

St. Peter's College,
April 4, 1883.

ON LINEAR DIFFERENTIAL EQUATIONS: IN PARTICULAR THAT SATISFIED BY THE SERIES

$$1+\frac{\alpha\beta\theta}{\gamma\varepsilon}x+\frac{\alpha.\alpha+1.\beta.\beta+1.\theta.\theta+1}{1.2.\gamma.\gamma+1.\varepsilon.\varepsilon+1}x^2+\ldots$$

By A. R. Forsyth, B.A., Fellow of Trinity College, Cambridge, and Professor of Mathematics in University College, Liverpool.

IN the present paper I have attempted the solution of several questions relating to the higher hypergeometric series F given by

$$1+\frac{\alpha\beta\theta}{\gamma\varepsilon}x+\frac{\alpha.\alpha+1.\beta.\beta+1.\theta.\theta+1}{1.2.\gamma.\gamma+1.\varepsilon.\varepsilon+1}x^2+\ldots$$

which are analogous to some of those arising in connection with the hypergeometric series of Gauss, and dealt with in the following papers:—

Gauss —*Gesammelte Werke*, bd. III., pp. 123–162, and 207–229, (1812).

Kummer —*Crelle*, t. XV. (1836), pp. 39–83, and 127–172.

Schwarz—*Crelle*, t. LXXV. (1873), pp. 292–335.

Cayley —*Camb. Phil. Trans.* t. XIII. (1880), "On the Schwarzian derivative and the Polyhedral Functions" (Part I.).

But the analytical properties alone have been kept in view and no attempt has been made to generalize the applications to geometry which occur in the above memoirs of Schwarz and Cayley and in others by Klein and Brioschi*; and there is no discussion of such relations as may arise when the elements α, β, θ, γ, ϵ are connected by one or more equations.

In this paper the following will be found to be the chief points:—

§§ 1–2: The series, its differential equation and two other particular integrals of the latter.

§§ 3–5: Two sets of relations (containing 34 in all) between F and contiguous functions, *i.e.* functions in which one element alone differs by unity from the corresponding element in F.

§ 6: An important transcendental series which is a particular case of the hypergeometric series.

§§ 7–10: Some summation-formulæ for the hypergeometric series of both the first and the second order.

§§ 11–15: An expression for the general integral of a linear differential equation of the n^{th} order when $(n-1)$ integrals are given; in particular for the cubic when two integrals are given.

§§ 16: A particular equation of the third order treated by Lommel (*Math. Ann.* II.).

§ 17: Application to the general linear cubic of Kummer's method (l. c.) adopted for the general linear equation of the second order.

§§ 18–28: Deduction of twelve particular integrals (of a certain general form) of the differential equation of § 1; and

§ 29: The differential equation satisfied by the quotient of two particular integrals of the linear equation of the third order.

---

* For a list of the memoirs relating to these applications cf. the above memoir of Cayley's.

The Series.

1. Consider the function $F\left\{\begin{pmatrix}\alpha, \beta, \theta \\ \gamma, \varepsilon\end{pmatrix}, x\right\}$ as defined by the equation

$$F = 1 + \frac{\alpha\beta\theta}{\gamma\varepsilon}x + \frac{\alpha.\alpha+1.\beta.\beta+1.\theta.\theta+1}{1.2.\gamma.\gamma+1.\varepsilon.\varepsilon+1}x^2 + \ldots \quad \ldots(1),$$

the law of formation of the coefficients being similar to that which holds in the case of Gauss's hypergeometric series. The coefficient of x^r being denoted by A_r, we have

$$(1+r)(\varepsilon+r)(\gamma+r)A_{r+1} = (\alpha+r)(\beta+r)(\theta+r)A_r.$$

Let D denote the operation $x\frac{d}{dx}$ and consider the differential equation

$$\left[(D+\alpha)(D+\beta)(D+\theta) - \frac{1}{x}D(D+\gamma-1)(D+\varepsilon-1)\right]y = 0 \ldots (2).$$

A solution of this can be obtained in a series; say,

$$y = B_0x^\mu + B_1x^{\mu+1} + B_2x^{\mu+2} + \ldots;$$

then to determine μ we have

$$-\mu(\mu+\gamma-1)(\mu+\varepsilon-1) = 0,$$

while the successive coefficients are connected by the relation

$$(\mu+r+1)(\mu+r+\gamma)(\mu+r+\varepsilon)B_{r+1} - (\mu+r+\alpha)(\mu+r+\beta)(\mu+r+\theta)B_r = 0.$$

Choosing $\mu = 0$ as one solution, and taking $B_0 = 1$, we have as the series

$$1 + \frac{\alpha\beta\theta}{\gamma\varepsilon}x + \frac{\alpha.\alpha+1.\beta.\beta+1.\theta.\theta+1}{1.2.\gamma.\gamma+1.\varepsilon.\varepsilon+1}x^2 + \ldots,$$

or the series (1) is a solution of the equation (2). Now the latter is

$$(1-x)D^3 + \{\gamma+\varepsilon-2-x(\alpha+\beta+\theta)\}D^2 + \{\gamma-1.\varepsilon-1.-x(\alpha\beta+\beta\theta+\theta\alpha)\}D - \alpha\beta\theta x = 0,$$

or, if d stand for $\frac{d}{dx}$,

$$D = xd,$$

$$D^2 = x^2d^2 + xd,$$

$$D^3 = x^3d^3 + 3x^2d^2 + xd,$$

$$(1-x)\,x^3d^3 + \{(\gamma+\varepsilon+1)\,x^2 - (\alpha+\beta+\theta+3)\,x^3\}\,d^2$$
$$+ \{\gamma\varepsilon x - x^2\,(\alpha\beta+\beta\theta+\theta\alpha+\alpha+\beta+\theta+1)\}\,d - \alpha\beta\theta x = 0,$$

so that F satisfies the differential equation

$$(1-x)\,x^2\frac{d^3F}{dx^3} + \{\gamma+\varepsilon+1-(\alpha+\beta+\theta+3)\,x\}\,x\frac{d^2F}{dx^2}$$

$$+ \{\gamma\varepsilon - x\,(\alpha\beta+\beta\theta+\theta\alpha+\alpha+\beta+\theta+1)\}\frac{dF}{dx} - \alpha\beta\theta F = 0 \ldots (3).$$

2. But another solution of the equation which determines μ is

$$\mu = 1-\gamma,$$

and the relation between the successive coefficients becomes

$$(r+1)\,(2-\gamma+r)\,(1+\varepsilon-\gamma+r)\,B_{r+1}$$
$$-(1+\alpha-\gamma+r)\,(1+\beta-\gamma+r)\,(1+\theta-\gamma+r)\,B_r = 0;$$

so that, choosing $B_0 = 1$, we have as another solution of (3)

$$x^{1-\gamma}F\left\{\begin{pmatrix}\alpha+1-\gamma,\ \beta+1-\gamma,\ \theta+1-\gamma\\ 2-\gamma,\ \varepsilon+1-\gamma\end{pmatrix},\ x\right\}.$$

Similarly, choosing $\mu = 1-\varepsilon$, a third solution is

$$x^{1-\varepsilon}F\left\{\begin{pmatrix}\alpha+1-\varepsilon,\ \beta+1-\varepsilon,\ \theta+1-\varepsilon\\ \gamma+1-\varepsilon,\ \ 2-\varepsilon\end{pmatrix},\ x\right\}.$$

Calling the two latter solutions

$$x^{1-\gamma}F_\gamma \text{ and } x^{1-\varepsilon}F_\varepsilon$$

respectively, we have three particular solutions, viz., F, $x^{1-\gamma}F_\gamma$, and $x^{1-\varepsilon}F_\varepsilon$, while the general solution is

$$y = AF + Bx^{1-\gamma}F_\gamma + Cx^{1-\varepsilon}F_\varepsilon,$$

A, B, and C being arbitrary constants.

It may be noticed that the more general equation

$$(y-a)\left\{(y-a)(y-b)\frac{d^3P}{dy^3}+(Ay+B)\frac{d^2P}{dy^2}\right\}$$
$$+(C+Gy)\frac{dP}{dy}+EP=0$$

can be reduced to the form (3) by means of the substitution

$$y-a=x(b-a)$$

and a corresponding determination of the elements $\alpha, \beta, \theta, \gamma, \varepsilon$ in terms of the constants in the equation.

The linear relations between F and the contiguous series.

3. Let the function $F\left\{\begin{pmatrix}\alpha+1, & \beta, & \theta\\ & \gamma, & \varepsilon\end{pmatrix}, x\right\}$ be denoted by $F_{\alpha+}$ and the function $F\left\{\begin{pmatrix}\alpha-1, & \beta, & \theta\\ & \gamma, & \varepsilon\end{pmatrix}, x\right\}$ by $F_{\alpha-}$. Such functions as these we may, after Gauss, call 'contiguous'; and there are obviously ten functions contiguous to a given one. Relations between these and the given one are separable into two sets; first, those in which only two functions besides the primary F (supposed present in every relation) occur; second, those in which three functions besides the primary occur. It will be noticed that, as written, the latter set is the only one into which the variable x enters as a coefficient.

Set I.

$$\left.\begin{aligned}
&(\varepsilon-\gamma)F+(\gamma-1)F_{\gamma-}-(\varepsilon-1)F_{\varepsilon-}=0\\
&(\beta-\alpha)F+\alpha F_{\alpha+}-\beta F_{\beta+}=0\\
&(\alpha-\theta)F+\theta F_{\theta+}-\alpha F_{\alpha+}=0\\
&(\theta-\beta)F+\beta F_{\beta+}-\theta F_{\theta+}=0
\end{aligned}\right\} G[2],$$

$$\left.\begin{aligned}
&(\gamma-\alpha-1)F+\alpha F_{\alpha+}-(\gamma-1)F_{\gamma-}=0\\
&(\gamma-\beta-1)F+\beta F_{\beta+}-(\gamma-1)F_{\gamma-}=0\\
&(\gamma-\theta-1)F+\theta F_{\theta+}-(\gamma-1)F_{\gamma-}=0\\
&(\varepsilon-\alpha-1)F+\alpha F_{\alpha+}-(\varepsilon-1)F_{\varepsilon-}=0\\
&(\varepsilon-\beta-1)F+\beta F_{\beta+}-(\varepsilon-1)F_{\varepsilon-}=0\\
&(\varepsilon-\theta-1)F+\theta F_{\theta+}-(\varepsilon-1)F_{\varepsilon-}=0
\end{aligned}\right\} G[5], [12].$$

Set II.

$$
\left.\begin{aligned}
&[(\gamma-\alpha)(\varepsilon-\alpha)-\alpha(\theta-\alpha)+(\theta-\alpha)(\alpha-\beta)x]F+\alpha(\theta-\alpha)(1-x)F_{\alpha+}-(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}\\
&\qquad+\frac{\theta-\varepsilon.\theta-\gamma.\alpha-\beta}{\theta-\beta}(F+F_{\theta-})=0\\
&[(\gamma-\alpha)(\varepsilon-\alpha)-\alpha(\beta-\alpha)+(\beta-\alpha)(\alpha-\theta)x]F+\alpha(\beta-\alpha)(1-x)F_{\alpha+}-(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}\\
&\qquad+\frac{\beta-\varepsilon.\beta-\gamma.\alpha-\theta}{\beta-\theta}(F+F_{\beta-})=0\\
&[(\gamma-\beta)(\varepsilon-\beta)-\beta(\theta-\beta)+(\theta-\beta)(\beta-\alpha)x]F+\beta(\theta-\beta)(1-x)F_{\beta+}-(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}\\
&\qquad+\frac{\theta-\varepsilon.\theta-\gamma.\beta-\alpha}{\theta-\alpha}(F+F_{\theta-})=0\\
&[(\gamma-\beta)(\varepsilon-\beta)-\beta(\alpha-\beta)+(\alpha-\beta)(\beta-\theta)x]F+\beta(\alpha-\beta)(1-x)F_{\beta+}-(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}\\
&\qquad+\frac{\alpha-\varepsilon.\alpha-\gamma.\beta-\theta}{\alpha-\theta}(F+F_{\alpha-})=0\\
&[(\gamma-\theta)(\varepsilon-\theta)-\theta(\alpha-\theta)+(\alpha-\theta)(\theta-\beta)x]F+\theta(\alpha-\theta)(1-x)F_{\theta+}-(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}\\
&\qquad+\frac{\alpha-\varepsilon.\alpha-\gamma.\theta-\beta}{\alpha-\beta}(F+F_{\alpha-})=0\\
&[(\gamma-\theta)(\varepsilon-\theta)-\theta(\beta-\theta)+(\beta-\theta)(\theta-\alpha)x]F+\theta(\beta-\theta)(1-x)F_{\theta+}-(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}\\
&\qquad+\frac{\beta-\varepsilon.\beta-\gamma.\theta-\alpha}{\beta-\alpha}(F+F_{\beta-})=0
\end{aligned}\right\}\quad G[1],\ [10].
$$

$$\left.\begin{aligned}
&(\gamma-\varepsilon)[\alpha+\varepsilon\gamma(\beta+\theta-\gamma-\varepsilon)x]F+\alpha\varepsilon\gamma(\varepsilon-\gamma)(1-x)F_{\alpha+}\\
&\qquad-\varepsilon(\alpha-\gamma)(\beta-\gamma)(\theta-\gamma)xF_{\gamma+}+\gamma(\alpha-\varepsilon)(\beta-\varepsilon)(\theta-\varepsilon)xF_{\varepsilon+}=0\\
&(\gamma-\varepsilon)[\beta+\varepsilon\gamma(\alpha+\theta-\gamma-\varepsilon)x]F+\beta\varepsilon\gamma(\varepsilon-\gamma)(1-x)F_{\beta+}\\
&\qquad-\varepsilon(\alpha-\gamma)(\beta-\gamma)(\theta-\gamma)xF_{\gamma+}+\gamma(\alpha-\varepsilon)(\beta-\varepsilon)(\theta-\varepsilon)xF_{\varepsilon+}=0\\
&(\gamma-\varepsilon)[\theta+\varepsilon\gamma(\alpha+\beta-\gamma-\varepsilon)x]F+\theta\varepsilon\gamma(\varepsilon-\gamma)(1-x)F_{\theta+}\\
&\qquad-\varepsilon(\alpha-\gamma)(\beta-\gamma)(\theta-\gamma)xF_{\gamma+}+\gamma(\alpha-\varepsilon)(\beta-\varepsilon)(\theta-\varepsilon)xF_{\varepsilon+}=0
\end{aligned}\right\}\quad G[4],\ [11].$$

$$\left.\begin{aligned}
&\gamma\varepsilon(\gamma-\varepsilon)[\gamma-1-(2\gamma+\varepsilon-\alpha-\beta-\theta-1)x]F-\gamma\varepsilon(\gamma-\varepsilon)(\gamma-1)(1-x)F_{\gamma-}\\
&\qquad-\varepsilon(\alpha-\gamma)(\beta-\gamma)(\theta-\gamma)xF_{\gamma+}+\gamma(\alpha-\varepsilon)(\beta-\varepsilon)(\theta-\varepsilon)xF_{\varepsilon+}=0\\
&\gamma\varepsilon(\varepsilon-\gamma)[\varepsilon-1-(2\varepsilon+\gamma-\alpha-\beta-\theta-1)x]F-\gamma\varepsilon(\varepsilon-\gamma)(\varepsilon-1)(1-x)F_{\varepsilon-}\\
&\qquad+\varepsilon(\alpha-\gamma)(\beta-\gamma)(\theta-\gamma)xF_{\gamma+}-\gamma(\alpha-\varepsilon)(\beta-\varepsilon)(\theta-\varepsilon)xF_{\varepsilon+}=0
\end{aligned}\right\}\quad G[15].$$

$$\left.\begin{aligned}
&(\alpha-\theta)(\theta-\beta)(\beta-\alpha)(1-x)F+(\gamma-\alpha)(\varepsilon-\alpha)(\theta-\beta)F_{\alpha-}\\
&\qquad+(\gamma-\beta)(\varepsilon-\beta)(\alpha-\theta)F_{\beta-}+(\gamma-\theta)(\varepsilon-\theta)(\beta-\alpha)F_{\theta-}=0
\end{aligned}\right\}\quad G[7].$$

$$\left.\begin{aligned}
&(\alpha-\beta)[\alpha+\beta-\varepsilon-1+(\theta-\gamma+1)x]F-(\alpha-\beta)(\gamma-1)(1-x)F_{\gamma}+(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}+(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}=0\\
&(\beta-\theta)[\beta+\theta-\varepsilon-1+(\alpha-\gamma+1)x]F-(\beta-\theta)(\gamma-1)(1-x)F_{\gamma-}+(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}+(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}=0\\
&(\theta-\alpha)[\theta+\alpha-\varepsilon-1+(\beta-\gamma+1)x]F-(\theta-\alpha)(\gamma-1)(1-x)F_{\gamma-}+(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}+(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}=0\\
&(\alpha-\beta)[\alpha+\beta-\gamma-1+(\theta-\varepsilon+1)x]F-(\alpha-\beta)(\varepsilon-1)(1-x)F_{\varepsilon-}+(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}+(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}=0\\
&(\beta-\theta)[\beta+\theta-\gamma-1+(\alpha-\varepsilon+1)x]F-(\beta-\theta)(\varepsilon-1)(1-x)F_{\varepsilon-}+(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}+(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}=0\\
&(\theta-\alpha)[\theta+\alpha-\gamma-1+(\beta-\varepsilon+1)x]F-(\theta-\alpha)(\varepsilon-1)(1-x)F_{\varepsilon-}+(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}+(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}=0
\end{aligned}\right\}\quad G[9],\ [14].$$

$$\left.\begin{aligned}
&(\theta-\beta)(\theta+\alpha+\beta-\gamma-\varepsilon)F+\alpha(\beta-\theta)(1-x)F_{\alpha+}+(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}-(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}=0\\
&(\beta-\alpha)(\theta+\alpha+\beta-\gamma-\varepsilon)F+\theta(\alpha-\beta)(1-x)F_{\theta+}+(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}-(\gamma-\beta)(\varepsilon-\beta)F_{\beta-}=0\\
&(\alpha-\theta)(\theta+\alpha+\beta-\gamma-\varepsilon)F+\beta(\theta-\alpha)(1-x)F_{\beta+}+(\gamma-\theta)(\varepsilon-\theta)F_{\theta-}-(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-}=0
\end{aligned}\right\}\quad G[3],\ [6].$$

$$\left.\begin{aligned}
&\gamma\varepsilon(\varepsilon-\gamma)(1-x)F-\gamma\varepsilon(\varepsilon-\gamma)F_{\alpha-}-\varepsilon(\beta-\gamma)(\theta-\gamma)xF_{\gamma+}+\gamma(\beta-\varepsilon)(\theta-\varepsilon)xF_{\varepsilon+}=0\\
&\gamma\varepsilon(\varepsilon-\gamma)(1-x)F-\gamma\varepsilon(\varepsilon-\gamma)F_{\beta-}-\varepsilon(\theta-\gamma)(\alpha-\gamma)xF_{\gamma+}+\gamma(\theta-\varepsilon)(\alpha-\varepsilon)xF_{\varepsilon+}=0\\
&\gamma\varepsilon(\varepsilon-\gamma)(1-x)F-\gamma\varepsilon(\varepsilon-\gamma)F_{\theta-}-\varepsilon(\alpha-\gamma)(\beta-\gamma)xF_{\gamma+}+\gamma(\alpha-\varepsilon)(\beta-\varepsilon)xF_{\varepsilon+}=0
\end{aligned}\right\}\quad G[8],\ [13].$$

In the above the letter G followed by numbers implies that the bracketed group corresponds with the equations belonging to those numbers in the set relating to the hypergeometric series given by Gauss, *Ges. Werke*, t. iii. p. 130.

4. The method by which the above were obtained was as follows. The coefficient in F of x^{m+1} is

$$\frac{\alpha(\alpha+1)\ldots(\alpha+m)\,\beta(\beta+1)\ldots(\beta+m)\,\theta(\theta+1)\ldots(\theta+m)}{1.2.3\ldots(1+m)\,\gamma(\gamma+1)\ldots(\gamma+m)\,\varepsilon(\varepsilon+1)\ldots(\varepsilon+m)},$$

or, say, M. Then the coefficient of the same power of the variable

$$\text{in } F_{\alpha-} \text{ is } M\frac{\alpha-1}{m+\alpha}; \quad \text{in } F_{\alpha+} \text{ is } M\frac{m+\alpha+1}{\alpha};$$

$$\ldots F_{\beta-} \ldots M\frac{\beta-1}{m+\beta}; \quad \ldots F_{\beta+} \ldots M\frac{m+\beta+1}{\beta};$$

$$\ldots F_{\theta-} \ldots M\frac{\theta-1}{m+\theta}; \quad \ldots F_{\theta+} \ldots M\frac{m+\theta+1}{\theta};$$

$$\ldots F_{\gamma-} \ldots M\frac{m+\gamma}{\gamma-1}; \quad \ldots F_{\gamma+} \ldots M\frac{\gamma}{m+\gamma+1};$$

$$\ldots F_{\varepsilon-} \ldots M\frac{m+\varepsilon}{\varepsilon-1}; \quad \ldots F_{\varepsilon+} \ldots M\frac{\varepsilon}{m+\varepsilon+1};$$

$$\text{in } xF_{\alpha+} \text{ it is } \frac{M}{\alpha}\,\frac{(m+1)(m+\gamma)(m+\varepsilon)}{(m+\beta)(m+\theta)};$$

$$\ldots xF_{\beta+} \ldots \frac{M}{\beta}\,\frac{(m+1)(m+\gamma)(m+\varepsilon)}{(m+\alpha)(m+\theta)};$$

$$\ldots xF_{\theta+} \ldots \frac{M}{\theta}\,\frac{(m+1)(m+\gamma)(m+\varepsilon)}{(m+\alpha)(m+\beta)};$$

$$\ldots xF_{\gamma+} \ldots M\gamma\,\frac{(m+1)(m+\varepsilon)}{(m+\alpha)(m+\beta)(m+\theta)};$$

$$\ldots xF_{\varepsilon+} \ldots M\varepsilon\,\frac{(m+1)(m+\gamma)}{(m+\alpha)(m+\beta)(m+\theta)};$$

$$\ldots xF_{\gamma-} \ldots \frac{M}{\gamma-1}\,\frac{(m+1)(m+\varepsilon)(m+\gamma)(m+\gamma-1)}{(m+\alpha)(m+\beta)(m+\theta)};$$

$$\ldots xF_{\varepsilon-} \ldots \frac{M}{\varepsilon-1}\,\frac{(m+1)(m+\gamma)(m+\varepsilon)(m+\varepsilon-1)}{(m+\alpha)(m+\beta)(m+\theta)};$$

$$\text{and } \ldots xF \ldots M\,\frac{(m+1)(m+\gamma)(m+\varepsilon)}{(m+\alpha)(m+\beta)(m+\theta)}.$$

Then if we take any one of the relations, say the first of the bracket corresponding to $G[8]$, $[13]$, it is of the form

$$(A - Bx)F - CF_{\alpha-} + ExF_{\gamma+} - GxF_{\varepsilon+} = 0.$$

If the term in x^{m+1} disappears its coefficient is zero and we have

$$A(m+\alpha)(m+\beta)(m+\theta) - B(m+1)(m+\gamma)(m+\varepsilon)$$
$$-C(\alpha+1)(m+\beta)(m+\theta)+E\gamma(m+1)(m+\varepsilon)-G\varepsilon(m+1)(m+\gamma)=0.$$

It is not difficult to prove that this is satisfied for all values of m provided

$$A = B = C,$$
$$A(\beta-\gamma)(\theta-\gamma) + E\gamma(\varepsilon-\gamma) = 0,$$
$$A(\beta-\varepsilon)(\theta-\varepsilon) - G\varepsilon(\gamma-\varepsilon) = 0,$$

so that taking $A = \gamma\varepsilon(\varepsilon-\gamma)$ we obtain the coefficients as given. When any one relation has been obtained the others of the set to which it belongs are written down by permuting the parameters, α, β and θ being permutable and likewise γ and ε.

Numerous formulæ can be obtained connecting the functions F_+, F_-, used in the above sets, either by eliminating F or by proceeding *ab initio*; but the functions, though all contiguous to F, are never contiguous one to another, and the resulting relations are those which subsist between functions, two of whose parameters differ by $+1$ or -1 from those of others. Thus it is easy to prove that

$$(\gamma-\alpha)(\varepsilon-\alpha)F_{\alpha-} + (\gamma-\beta)(\varepsilon-\beta)F_{\beta-} + (\gamma-\theta)(\varepsilon-\theta)F_{\theta-} = 0,$$

and by means of this each equation of the set $G[9]$, $[14]$ can be transformed into one in which only three functions will enter. Again

$$\gamma(\alpha-\varepsilon)(\beta-\theta)F_{\alpha-} + \gamma(\beta-\varepsilon)(\theta-\alpha)F_{\beta-} + \gamma(\theta-\varepsilon)(\alpha-\beta)F_{\theta-}$$
$$+ (\alpha-\beta)(\beta-\theta)(\theta-\alpha)xF_{\gamma+} = 0.$$

5. The above formulæ can be generalised to the case of the hypergeometric series of the n^{th} order. Denoting the parameters by $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_2, \beta_3, \ldots, \beta_n$, so that

$$F = 1 + \frac{\alpha_1.\alpha_2\ldots\alpha_n}{\beta_2\ldots\beta_n}x + \frac{\alpha_1.\alpha_1+1.\alpha_2.\alpha_2+1\ldots\alpha_n.\alpha_n+1}{1.2.\beta_2.\beta_2+1\ldots\ldots\beta_n.\beta_n+1}x^2+\ldots,$$

and writing, after Sylvester,

$\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, .., \alpha_n) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots(\alpha_1 - \alpha_n)(\alpha_2 - \alpha_3)\ldots(\alpha_{n-1} - \alpha_n)$,

we can easily obtain

$$\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, .., \alpha_n)(1-x)F$$
$$+ \sum_{r=1}^{r=n} \{(-1)^r \zeta^{\frac{1}{2}}(\alpha_1, .., \alpha_{r-1}, \alpha_{r+1}, .., \alpha_n) \prod_{s=1}^{s=n-1} (\alpha_r - \beta_s) F_{\alpha_r -}\} = 0,$$

which corresponds to *G* [7].

6. Particular cases of the hypergeometric series occur in transcendent series which have already been discussed in various papers. The general term in the preceding series *F* may be put into the form

$$\left\{\prod_{r=1}^{r=n} \frac{\alpha_r . \alpha_r + 1 \ldots \alpha_r + s}{\beta_r . \beta_r + 1 \ldots \beta_r + s}\right\} x^{s+1},$$

in which $\beta_1 = 1$.

Let $\alpha_1 = \beta_1 = 1$; $\alpha_r = \beta_r - 1 = \theta - 1$ say, where θ is the same for all; then the general term in *F* is

$$\left(\frac{\theta - 1}{\theta + s}\right)^{n-1} x^{s+1},$$

or in the hypergeometric series of the $(n+1)^{\text{th}}$ order denoted, with the above suppositions, by $F_{n+1}{}^{(\theta)}$ we have

$$F^{(\theta)}{}_{n+1} = 1 + \left(\frac{\theta - 1}{\theta}\right)^x x + \left(\frac{\theta - 1}{\theta + 1}\right)^n x^2 + \ldots + \left(\frac{\theta - 1}{\theta + s}\right)^n x^{s+1} + \ldots .$$

Writing $\theta = 2$ and multiplying throughout by x we have

$$x F^{(2)}{}_{n+1} = \sum_{r=1}^{r=\infty} \frac{x^r}{r^n} .$$

The series on the right-hand side has been discussed, for the case of $n = 2$, by Abel, *Œuvres*, t. II, p. 189; see also for other references a paper by Mr. Glaisher, in *The Messenger of Mathematics* (1872).

Let $D \equiv x \dfrac{d}{dx}$ and denote the general series by F_{n+1}; then the differential equation satisfied by it is

$$\left[\prod_{r=1}^{r=n+1} (D + \alpha_r) - \frac{1}{x} D \prod_{r=2}^{r=n+1} (D + \beta_r - 1)\right] y = 0,$$

the other integrals of which are of the form

$$x^{1-\beta_r} F_{\beta_r}.$$

As a particular case we have

$$\left[(D+1)^{n+1} - \frac{1}{x} D\,(D+1)^n\right] F_{n+1}^{(2)} = 0;$$

or, what is the same thing,

$$[(1-x)\,(D+1)^{n+1} - (D+1)^n]\, F_{n+1}^{(2)} = 0.$$

Again, $$\left[(D+\theta-1)^{n+1} - \frac{1}{x}(D+\theta-1)^n\right] F_{n+1}^{(\theta)} = 0;$$

i.e. $$[(1-x)\,(D+\theta-1)^{n+1} - (\theta-1)\,(D+\theta-1)^n]\, F_{n+1}^{(\theta)} = 0.$$

In the case when θ is an integer other than 2, we have as an obvious relation,

$$xF_{n+1}^{(2)} - \frac{x^{\theta-1}}{(\theta-1)^n} F_{n+1}^{(\theta)} = x + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \ldots + \frac{x^{\theta-2}}{(\theta-2)^n}.$$

Again, if θ, θ' be both integers and $\theta' > \theta$,

$$xF_{n+1}^{(2)} - \frac{x^{\theta'-1}}{(\theta'-1)^n} F_{n+1}^{(\theta')} = x + \frac{x^2}{2^n} + \ldots + \frac{x^{\theta'-2}}{(\theta'-2)^n},$$

and therefore

$$\frac{x^{\theta-1}}{(\theta-1)^n} F_{n+1}^{(\theta)} - \frac{x^{\theta'-1}}{(\theta'-1)^n} F_{n+1}^{(\theta')} = \frac{x^{\theta-1}}{(\theta-1)^n} + \frac{x^{\theta}}{\theta^n} + \ldots + \frac{x^{\theta'-2}}{(\theta'-2)^n},$$

so that the series for $F_{n+1}^{(2)}$ can be separated into parts, each of which is the difference of two expressions connected with series of the form $F_{n+1}^{(\theta)}$.

Some Summation-Formulæ.

7. Let u be a function of two variables z and x expansible in ascending powers of both; and let v_n be the coefficient of x^n in such expansion, so that

$$u = \Sigma z^n v_n.$$

Then, considering the n^{th} term alone,

$$z\frac{du}{dz} = nu \ldots\ldots (\alpha), \qquad z^2\frac{d^2u}{dz^2} = n\,(n-1)\,u \ldots\ldots (\beta).$$

Let v_n be a solution of the differential equation

$$\frac{d^2v_n}{dx^2} + P\frac{dv_n}{dx} + Qv_n = 0,$$

so that our single term satisfies the equation

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

Now Q will, and P may, involve n; if between (α), (β), and (γ) n can be eliminated so as to leave the equation linear in u, then this resulting equation will be satisfied, not only by the single term, but also by the general value of u.

We may take two well-known examples.

Ex. 1. Let $Q = \dfrac{n(n+1)}{1-x^2}$, $P = -\dfrac{2x}{1-x^2}$; then the final equation is

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + z^2\frac{d^2u}{dz^2} + 2z\frac{du}{dz} = 0;$$

that is

$$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + z\frac{d^2}{dz^2}(zu) = 0,$$

which is the differential equation for a solid zonal harmonic, while the subsidiary equation is that for the surface zonal harmonic.

Ex. 2. Let $Q = 1 - \dfrac{n^2}{x^2}$ and $P = \dfrac{1}{x}$; then the final equation is

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + u = \frac{1}{x^2}\left[z^2\frac{d^2u}{dz^2} + z\frac{du}{dz}\right];$$

that is

$$x\frac{d}{dx}\left(x\frac{du}{dx}\right) + x^2u = z\frac{d}{dz}\left(z\frac{du}{dz}\right);$$

or, writing

$$x = e^{\xi}, \quad z = e^{\theta i},$$

this is

$$\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\theta^2} + ue^{2\xi} = 0.$$

Now the subsidiary equation is that satisfied by Bessel's functions; and $J_n(x)$ we know to be the coefficient of z^n in

$$e^{\frac{1}{2}x\left(z - \frac{1}{z}\right)}$$

so that a solution of the foregoing partial differential equation is

$$\log u = ie^{\xi}\sin\theta;$$

or, as the real and imaginary parts of u must separately satisfy the equation, a more general solution is given

$$u = A\cos(e^{\xi}\sin\theta + \beta).$$

8. The above can obviously be extended to the case of several variables, one arising for each parameter in the function defined by the subsidiary equation. Consider now the differential equation to the hypergeometric series $F(\alpha, \beta, \gamma, x)$ as treated by Gauss; it is

$$\frac{d^2y}{dx^2} + \frac{\gamma - (\alpha+\beta+1)\,x}{x(1-x)}\frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)}y = 0.$$

Let $$S = F(\alpha, \beta, \gamma, x)\, u^\alpha v^\beta x^\gamma,$$

so that we may substitute S for y in the above. Now

$$u\frac{dS}{du} = \alpha S, \quad v\frac{dS}{dv} = \beta S, \quad z\frac{dS}{dz} = \gamma S,$$

and

$$uv\frac{d^2S}{du\,dv} = \alpha\beta S.$$

Eliminating α, β and γ we have

$$x(1-x)\frac{d^2S}{dx^2} + z\frac{d^2S}{dx\,dz} - x\frac{dS}{dx} - ux\frac{d^2S}{du\,dx} - uv\frac{d^2S}{du\,dv} - vx\frac{d^2S}{dv\,dx} = 0,$$

and therefore the equation

$$\left(x\frac{d}{dx} + z\frac{d}{dz}\right)\frac{d\sigma}{dx} - x\frac{d}{dx}\,x\frac{d}{dx}\,\sigma = ux\frac{d^2\sigma}{du\,dx} + xv\frac{d^2\sigma}{dx\,dv} + vu\frac{d^2\sigma}{dv\,du}$$

is satisfied by

$$\sigma = \sum_{\alpha=1}^{\alpha=\infty}\sum_{\beta=1}^{\beta=\infty}\sum_{\gamma=1}^{\gamma=\infty} u^\alpha v^\beta z^\gamma F(\alpha, \beta, \gamma, x),$$

and we shall suppose u, v, z, x so small as to render the quadruply-infinite series convergent.

In σ take a term of the form $f_{r,p}(m)\, x^m v^p z^r$, where $f(m)$ is a function of u and m; substituting in the differential equation (which will be satisfied by this single term), we have

$$(m+1)(m+r)\,f_{r,p}(m+1) - m^2 f_{r,p}(m) = mu\frac{df(m)}{du} + pu\frac{df(m)}{du} + mpf(m),$$

or

$$f_{r,p}(m+1) = \frac{m+p}{(m+1)(m+r)}\left(m + u\frac{d}{du}\right)f_{r,p}(m);$$

and therefore

$$u^{m+1} f_{r,p}(m+1) = \frac{m+p}{(m+1)(m+r)} u^2 \frac{d}{du} \{u^m f_{r,p}(m)\}$$

$$= \frac{m+p\,!\; r-1\,!}{m+1\,!\; m+r\,!\; p-1\,!} \left(u^2 \frac{d}{du}\right)^{m+1} \{f_{r,p}(0)\}.$$

Now $f_{r,p}(0) =$ coefficient of $v^p z^r x^0$ in σ

$$= u + u^2 + \ldots = \frac{1}{1-u} - 1,$$

and we have therefore to find

$$\left(u^2 \frac{d}{du}\right)^{m+1} \left\{\frac{1}{1-u} - 1\right\} = U \text{ say.}$$

Let $u = \frac{1}{t}$; therefore $\frac{du}{u^2} = -dt$ and

$$\frac{1}{1-u} = \frac{t}{t-1} = 1 - \frac{1}{1-t},$$

so that

$$U = (-1)^{m+1} \frac{d^{m+1}}{dt^{m+1}} \left\{-\frac{1}{1-t}\right\}$$

$$= (-1)^m \frac{m+1\,!}{(1-t)^{m+2}} = m+1\,! \; \frac{u^{m+2}}{(1-u)^{m+2}},$$

and therefore

$$u^{m+1} f_{r,p}(m+1) = \frac{m+p\,!\; r-1\,!}{m+r\,!\; p-1\,!} \frac{u^{m+2}}{(1-u)^{m+2}},$$

or

$$f_{r,p}(m) = \frac{p+m-1\,!\; r-1\,!}{r+m-1\,!\; p-1\,!} \frac{u}{(1-u)^{m+1}}.$$

Similarly, if we take in σ a term of the form $\phi_{r,n}(m)\, x^m u^n z^r$, we can prove that

$$\phi_{r,n}(m) = \frac{n+m-1\,!\; r-1\,!}{r+m-1\,!\; n-1\,!} \frac{v}{(1-v)^{m+1}}.$$

Now take in σ a term of the form $\psi_{n,p}(m)\, x^m u^n v^p$, where ψ is a function of z. Then, substituting, we obtain

$$(m+1)\left(m+z\frac{d}{dz}\right)\psi_{n,p}(m+1) - m^2\psi_{n,p}(m) = (mn+np+pm)\,\psi_{n,p}(m),$$

and therefore

$$\left(m+z\frac{d}{dz}\right)\psi_{n,p}(m+1) = \frac{m+p.m+n}{m+1}\,\psi_{n,p}(m);$$

that is,
$$\frac{d}{dz}\{z^m\psi_{n,p}(m+1)\} = \frac{m+p.m+n}{m+1}\,z^{m-1}\psi_{n,p}(m),$$

so that

$$\frac{d^{m+1}}{dz^{m+1}}\{z^m\psi_{n,p}(m+1)\} = \frac{m+p\,!\;m+n\,!}{m+1\,!\;p-1\,!\;n-1\,!}\,z^{-1}\psi_{n,p}(0).$$

Now
$$\psi_{n,p}(0) = \text{coefficient of } u^n v^p x^0 \text{ in } \sigma$$
$$= z + z^2 + z^3 + \ldots$$
$$= \frac{z}{1-z};$$

and therefore

$$\frac{d^{m+1}}{dz^{m+1}}\{z^m\psi_{n,p}(m+1)\} = \frac{m+p\,!\;m+n\,!}{m+1\,!\;p-1\,!\;n-1\,!}\,\frac{1}{1-z},$$

or

$$\frac{d^m}{dz^m}\{z^{m-1}\,\psi_{n,p}(m)\} = \frac{m+p-1\,!\;m+n-1\,!}{m\,!\;p-1\,!\;n-1\,!}\{1+z+z^2\ldots\}.$$

Integrate each side m times; from the nature of the coefficients and from the fact that z^0 does not occur in σ no constants will be necessary, and therefore

$$\frac{z^{m-1}.\psi_{n,p}(m).p-1\,!\;n-1\,!\;m\,!}{m+p-1\,!\;m+n-1\,!} = \frac{z^m}{m\,!} + \frac{z^{m+1}}{m+1\,!} + \frac{z^{m+2}}{m+2\,!}\,2\,! + \ldots$$
$$= \frac{z^m}{m\,!}\,F(1,\,1,\,m+1,\,z),$$

and therefore

$$\psi_{n,p}(m) = \frac{m+p-1\,!\;m+n-1\,!}{m\,!\;p-1\,!\;m\,!\;n-1\,!}\, zF(1,\,1,\,m+1,\,z).$$

9. Combining our results we therefore have the equations

$$\sigma = \sum_{\alpha=1}^{\alpha=\infty}\sum_{\beta=1}^{\beta=\infty}\sum_{\gamma=1}^{\gamma=\infty} u^\alpha v^\beta z^\gamma F(\alpha,\,\beta,\,\gamma,\,x)$$

$$= \sum_{m=0}^{m=\infty}\sum_{p=1}^{p=\infty}\sum_{r=1}^{r=\infty} x^m v^p z^r \frac{p+m-1\,!\;r-1\,!}{r+m-1\,!\;p-1\,!}\,\frac{u}{(1-u)^{m+1}}$$

$$= \sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}\sum_{r=1}^{r=\infty} x^m v^n z^r \frac{n+m-1\,!\;r-1\,!}{r+m-1\,!\;p-1\,!}\,\frac{v}{(1-v)^{m+1}}$$

$$= \sum_{m=0}^{m=\infty}\sum_{n=1}^{n=\infty}\sum_{p=1}^{p=\infty} x^m u^n v^p \frac{p+m-1\,!\;n+m-1\,!}{m\,!\;p-1\,!\;m\,!\;n-1\,!}\, zF(1,1,m+1,z).$$

It is obvious that $\dfrac{\sigma}{uvz}$ contains only integral powers of u, v and z; so that if this factor uvz common to every term be removed from the equations formed by the right-hand side we have the relations slightly simplified as follows:—

$$\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=\infty}\sum_{r=0}^{r=\infty} u^n v^p z^r F(n+1,\,p+1,\,r+1,\,x)$$

$$= \sum_{m=0}^{m=\infty}\sum_{p=0}^{p=\infty}\sum_{r=0}^{r=\infty} x^m v^p z^r \frac{m+p\,!\;r\,!}{m+r\,!\;p\,!}\,\frac{1}{(1-u)^{m+1}}$$

$$= \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}\sum_{r=0}^{r=\infty} x^m u^n z^r \frac{m+n\,!\;r\,!}{m+r\,!\;n\,!}\,\frac{1}{(1-v)^{m+1}}$$

$$= \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=\infty} x^m u^n v^p \frac{m+p\,!\;m+n\,!}{m\,!\;p\,!\;m\,!\;n\,!}\, F(1,\,1,\,m+1,\,z).$$

10. Treating the hypergeometric series of the second order in an exactly similar manner we obtain the relations

$$\sum_{k=0}^{k=\infty}\sum_{p=0}^{p=\infty}\sum_{q=0}^{q=\infty}\sum_{r=0}^{r=\infty}\sum_{l=0}^{l=\infty} s^k v^p u^q z^r w^l \, F\left\{\begin{pmatrix} k+1, & q+1, & p+1 \\ r+1, & & l+1 \end{pmatrix}, \; x\right\}$$

$$\sum_{m=0}^{m=\infty}\sum_{p=0}^{p=\infty}\sum_{q=0}^{q=\infty}\sum_{r=0}^{r=\infty}\sum_{l=0}^{l=\infty} x^m v^p u^q z^r w^l \, \frac{m+p\,!\;m+q\,!\;r\,!\;l\,!}{m+r\,!\;m+l\,!\;p\,!\;q\,!} \, \frac{1}{(1-s)^{m+1}}$$

$$\sum_{m=0}^{m=\infty}\sum_{k=0}^{k=\infty}\sum_{q=0}^{q=\infty}\sum_{r=0}^{r=\infty}\sum_{l=0}^{l=\infty} x^m s^k u^q z^r w^l \, \frac{m+k\,!\;m+q\,!\;r\,!\;l\,!}{m+r\,!\;m+l\,!\;k\,!\;q\,!} \, \frac{1}{(1-v)^{m+1}}$$

$$\sum_{m=0}^{m=\infty}\sum_{k=0}^{k=\infty}\sum_{p=0}^{p=\infty}\sum_{r=0}^{r=\infty}\sum_{l=0}^{l=\infty} x^m s^k v^p z^r w^l \, \frac{m+k\,!\;m+p\,!\;r\,!\;l\,!}{m+r\,!\;m+l\,!\;k\,!\;p\,!} \, \frac{1}{(1-u)^{m+1}}$$

$$\sum_{m=0}^{m=\infty}\sum_{k=0}^{k=\infty}\sum_{p=0}^{p=\infty}\sum_{n=0}^{n=\infty}\sum_{l=0}^{l=\infty} x^m s^k v^p u^n w^l \, \frac{m+k\,!\;m+p\,!\;m+n\,!\;l\,!}{m\,!\;k\,!\;m\,!\;p\,!\;m+l\,!\;n\,!} \, F(1,\,1,\,m+1,$$

$$\sum_{m=0}^{m=\infty}\sum_{k=0}^{k=\infty}\sum_{p=0}^{p=\infty}\sum_{n=0}^{n=\infty}\sum_{r=0}^{r=\infty} x^m s^k v^p u^n z^r \, \frac{m+k\,!\;m+p\,!\;m+n\,!\;r\,!}{m\,!\;h\,!\;m\,!\;p\,!\;m+r\,!\;n\,!} \, F(1,\,1,\,m+1,$$

where, in the last two expressions, F denotes the hypergeometric series of the first order.

The generalisation to the case of a hypergeometric series of higher order is obvious.

Expression for the general integral of a linear differential equation of the n^{th} order, when $n-1$ particular integrals are given.

11. If one solution, say y_1, of the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

be known, then the general solution is given by

$$y = Cy_1 + Ay_1 \int \frac{e^{-\int P dx}}{y_1^{\,2}} \, dx \,.$$

Similarly, if two solutions y_1, y_2 of a linear differential equation of the third order

$$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0$$

be known an expression can be found for the general solution. For

$$\frac{d^3y_1}{dx^3} + P\frac{d^2y_1}{dx^2} + Q\frac{dy_1}{dx} + Ry_1 = 0,$$

$$\frac{d^3y_2}{dx^3} + P\frac{d^2y_2}{dx^2} + Q\frac{dy_2}{dx} + Ry_2 = 0.$$

Between these three equations eliminate Q, R, then

$$\begin{vmatrix} y''', & y', & y \\ y_1''', & y_1', & y_1 \\ y_2''', & y_2', & y_2 \end{vmatrix} + P \begin{vmatrix} y'', & y', & y \\ y_1'', & y_1', & y_1 \\ y_2'', & y_2', & y_2 \end{vmatrix} = 0,$$

the integral of which is

$$\begin{vmatrix} y'', & y', & y \\ y_1'', & y_1', & y_1 \\ y_2'', & y_2', & y_2 \end{vmatrix} = Ae^{-\int P dx} = T \text{ say};$$

that is,

$$\frac{d^2y}{dx^2}\left(y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right) + \frac{dy}{dx}\left(y_1\frac{d^2y_2}{dx^2} - y_2\frac{d^2y_1}{dx^2}\right) + y\left(\frac{d^2y_1}{dx^2}\frac{dy_2}{dx} - \frac{d^2y_2}{dx^2}\frac{dy_1}{dx}\right) = T.$$

Moreover, we have identically

$$\frac{d^2y_1}{dx^2}\left(y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right) + \frac{dy_1}{dx}\left(y_1\frac{d^2y_2}{dx^2} - y_2\frac{d^2y_1}{dx^2}\right) + y\left(\frac{d^2y_1}{dx^2}\frac{dy_2}{dx} - \frac{d^2y_2}{dx^2}\frac{dy_1}{dx}\right) = 0;$$

and therefore

$$\left(y_1\frac{d^2y}{dx^2} - y\frac{d^2y_1}{dx^2}\right)\left(y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right) - \left(y_1\frac{dy}{dx} - y\frac{dy_1}{dx}\right)\left(y_2\frac{d^2y_1}{dx^2} - y_1\frac{d^2y_2}{dx^2}\right) = Ty_1;$$

and the integration of this gives

$$\frac{y_1\dfrac{dy}{dx} - y\dfrac{dy_1}{dx}}{y_2\dfrac{dy_1}{dx} - y_1\dfrac{dy_2}{dx}} = -C + \int \frac{Ty_1}{\left(y_2\dfrac{dy_1}{dx} - y_1\dfrac{dy_2}{dx}\right)^2}dx,$$

or

$$y_1\frac{dy}{dx}-y\frac{dy_1}{dx}=-C\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)$$
$$+\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\int\frac{Ty_1dx}{\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)^2}.$$

Similarly, taking the identical relation obtained by putting $y=y_2$ and $T=0$, and treating it just as above, we obtain the result

$$y_2\frac{dy}{dx}-y\frac{dy_2}{dx}=B\left(y_2\frac{dy_1}{dx}-y_1\frac{dy}{dx}\right)$$
$$+\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\int\frac{Ty_2dx}{\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)^2}.$$

Multiplying the former of these by y_2 and subtracting from the latter multiplied by y_1, the factor $y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}$ disappears, and we have

$$y=By_1+Cy_2+y_1\int\frac{Ty_2dx}{\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)^2}-y_2\int\frac{Ty_1dx}{\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)^2};$$

or, restoring the value of T, we have the theorem:

If y_1, y_2 be two different particular integrals of the equation

$$\frac{d^3y}{dx^3}+P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0,$$

then the general solution of the equation is

$$y=By_1+Cy_2+A\left[y_1\int\frac{e^{-\int Pdx}y_2dx}{\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)^2}-y_2\int\frac{e^{-\int Pdx}y_1dx}{\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)^2}\right].$$

12. We are also able to solve, under these conditions, the equation

$$\frac{d^3y}{dx^3}+P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=S.$$

For, proceeding as in the former case, the first integral is

$$\begin{vmatrix} y'', & y', & y \\ y_1'', & y_1', & y_1 \\ y_2'', & y_2', & y_2 \end{vmatrix} = Ae^{-\int P dx} + e^{-\int P dx} \int S_1 e^{\int P dx} dx,$$

where $$S_1 = S(y_1'' y_2' - y_2'' y_1')$$

and the only difference will be that we must substitute for T, instead of $Ae^{-\int P dx}$, the right-hand side of the last equation.

13. As a particular example, take the equation satisfied by the higher hypergeometric series; in this case

$$\phi = \frac{\gamma + \varepsilon + 1 - (\alpha + \beta + \theta + 3)x}{x(1-x)}$$
$$= \frac{\gamma + \varepsilon + 1}{x} + \frac{\gamma + \varepsilon - \alpha - \beta - \theta - 2}{1-x};$$

and therefore

$$Ae^{-\int P dx} = \frac{A}{x^{\gamma+\varepsilon+1}(1-x)^{\gamma+\varepsilon-\alpha-\beta-\theta-2}} = T.$$

Now take

$$y = F,$$
$$y_1 = x^{1-\gamma} F_\gamma,$$
$$y_2 = x^{1-\varepsilon} F_\varepsilon,$$

and to obtain the value of A, consider the lowest power of x on the left-hand side; it arises from the term

$$y\left(\frac{d^2 y_1}{dx^2}\frac{dy_2}{dx} - \frac{d^2 y_2}{dx^2}\frac{dy_1}{dx}\right),$$

and is

$$x^{-\varepsilon-\gamma-1}[(1-\gamma.-\gamma.1-\varepsilon) - (1-\varepsilon.-\varepsilon.1-\gamma)],$$

that is $$x^{-\varepsilon-\gamma-1} 1-\gamma.1-\varepsilon.\varepsilon-\gamma,$$

and therefore $$A = 1-\gamma.1-\varepsilon.\varepsilon-\gamma.$$

We thus have

$$F = 1-\gamma.1-\varepsilon.\varepsilon-\gamma\left[x^{1-\gamma}F_\gamma \int \frac{(1-x)^{\alpha+\beta+\theta-\gamma-\varepsilon+2} F_\varepsilon dx}{x^{\gamma+2\varepsilon}.V^2}\right.$$
$$\left. - x^{1-\varepsilon}F_\varepsilon \int \frac{(1-x)^{\alpha+\beta+\theta-\gamma-\varepsilon+2} F_\gamma dx}{x^{2\gamma+\varepsilon}.V^2}\right],$$

where

$$V = x^{1-\epsilon-\gamma}(\epsilon-\gamma)F_\epsilon F_\gamma + x^{2-\epsilon-\gamma}\left(F_\epsilon \frac{dF_\gamma}{dx} - F_\gamma \frac{dF_\epsilon}{dx}\right).$$

14. If only one solution be given, say y_1, it is easy to depress the equation to the second degree. If we write

$$y = y_1 \int v dx,$$

then the differential equation for v is

$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(\frac{3}{y_1}\frac{dy_1}{dx} + P\right) + v\left(\frac{3}{y_1}\frac{d^2y_1}{dx^2} + \frac{2}{y_1}\frac{dy_1}{dx} + Q\right) = 0.$$

Or it may be depressed thus:—Since

$$\frac{d^3y_1}{dx^3} + P\frac{d^2y_1}{dx^2} + Q\frac{dy_1}{dx} + Ry_1 = 0$$

we have

$$y_1\frac{d^3y}{dx^3} - y\frac{d^3y_1}{dx^3} + P\left(y_1\frac{d^2y}{dx^2} - y\frac{d^2y_1}{dx^2}\right) + Q\left(y_1\frac{dy}{dx} - y\frac{dy_1}{dx}\right) = 0.$$

Let $$U = y_1\frac{dy}{dx} - y\frac{dy_1}{dx},$$

then $$\frac{dU}{dx} = y_1\frac{d^2y}{dx^2} - y\frac{d^2y_1}{dx^2},$$

$$\frac{d^2U}{dx^2} = y_1\frac{d^3y}{dx^3} - y\frac{d^3y_1}{dx^3} + \frac{dy_1}{dx}\frac{d^2y}{dx^2} - \frac{dy}{dx}\frac{d^2y_1}{dx^2}$$

$$= y_1\frac{d^3y}{dx^3} - y\frac{d^3y_1}{dx^3} + \frac{1}{y_1}\left(\frac{dy_1}{dx}\frac{dU}{dx} - \frac{d^2y_1}{dx^2}U\right);$$

and therefore the equation for U is

$$\frac{d^2U}{dx^2} + \left(P - \frac{1}{y_1}\frac{dy_1}{dx}\right)\frac{dU}{dx} + \left(Q + \frac{1}{y_1}\frac{d^2y_1}{dx^2}\right)U = 0.$$

Each of these is of the form

$$\frac{d^2y}{dx^2} + P'\frac{dy}{dx} + Q'y = 0,$$

but with the general values of P' and Q' the integration cannot proceed further.

15. The theorem enunciated for the linear differential equation of the third order can be generalized to the case of the linear differential equation of the n^{th} order, say

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n y = 0.$$

Let there be given $n-1$ different particular integrals $y_1, y_2, \ldots, y_{n-1}$ of this; then we have

$$\frac{d^n y_\mu}{dx^n} + P_1 \frac{d^{n-1} y_\mu}{dx^{n+1}} + \ldots + P_{n-1} \frac{dy_\mu}{dx} + P_n y_\mu = 0,$$

for $\mu = 1, 2, \ldots, n-1.$

Eliminating $P_2, P_3, \ldots, P_n$, we have

$$\begin{vmatrix} \frac{d^n y}{dx^n}, & \frac{d^{n-2} y}{dx^{n-2}}, & \frac{d^{n-3} y}{dx^{n-3}}, \ldots, y \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \frac{d^n y_\mu}{dx^n}, & \frac{d^{n-2} y_\mu}{dx^{n-2}}, \ldots\ldots, y_\mu \\ \ldots\ldots\ldots\ldots \end{vmatrix} + P_1 \begin{vmatrix} \frac{d^{n-1} y}{dx^{n-1}}, & \frac{d^{n-2} y}{dx^{n-2}}, \ldots, y \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \frac{d^{n-1} y_\mu}{dx^{n-1}}, & \frac{d^{n-2} y_\mu}{dx^{n-2}}, \ldots, y_\mu \\ \ldots\ldots\ldots\ldots \end{vmatrix} = 0;$$

or, integrating,

$$\begin{vmatrix} \frac{d^{n-1} y}{dx^{n-1}}, & \frac{d^{n-2} y}{dx^{n-2}}, \ldots, y \\ \ldots\ldots\ldots\ldots \\ \frac{d^{n-1} y_\mu}{dx^{n-1}}, & \frac{d^{n-2} y_\mu}{dx^{n-2}}, \ldots, y_\mu \\ \ldots\ldots\ldots\ldots \\ \frac{d^{n-1} y_{n-1}}{dx^{n-1}}, & \frac{d^{n-2} y_{n-1}}{dx^{n-2}}, \ldots, y_{n-1} \end{vmatrix} = A e^{-\int P_1 dx} = T \text{ say},$$

while the determinant vanishes if $y = y_1, y_2, \ldots, y_{n-1}$.

Call this equation in y

$$A_0 \frac{d^{n-1} y}{dx^{n-1}} + A_1 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + A_{n-2} \frac{dy}{dx} + A_{n-1} y = T \ldots\ldots (1).$$

Then we have

$$A_0 \frac{d^{n-1} y_\mu}{dx^{n-1}} + A_1 \frac{d^{n-2} y_\mu}{dx^{n-2}} + \ldots + A_{n-2} \frac{dy_\mu}{dx} + A_{n-1} y_\mu = 0 \ldots (2),$$

for $\mu = 1, 2, \ldots, n-1.$

For shortness, let $D = \frac{d}{dx}$, and write

$$\left|\begin{array}{lll} D^{n-2}y_1, & D^{n-3}y_1, \ldots, & y_1 \\ D^{n-2}y_2, & D^{n-3}y_2, \ldots, & y_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \vdots & \text{none omitted} & \vdots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^{n-2}y_{n-1}, & D^{n-3}y_{n-1}, \ldots, & y_{n-1} \end{array}\right| = Q \ldots\ldots\ldots \text{(III)},$$

then

$$\left|\begin{array}{lll} D^{n-1}y_1, & D^{n-3}y_1, \ldots, & y_1 \\ D^{n-1}y_2, & D^{n-3}y_2, \ldots, & y_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \vdots & \text{none omitted} & \vdots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^{n-1}y_{n-1}, & D^{n-3}y_{n-1}, \ldots, & y_{n-1} \end{array}\right| = \frac{dQ}{dx};$$

while, if

$$\left|\begin{array}{lll} D^{n-2}y, & D^{n-3}y, \ldots, & y \\ D^{n-2}y_1, & D^{n-3}y_1, \ldots, & y_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \vdots & y_m \text{ omitted} & \vdots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^{n-2}y_{n-1}, & D^{n-3}y_{n-1}, \ldots, & y_{n-1} \end{array}\right| = R_m$$

then

$$\left|\begin{array}{lll} D^{n-1}y, & D^{n-3}y, \ldots, & y \\ D^{n-1}y_1, & D^{n-3}y_1, \ldots, & y_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \vdots & y_m \text{ omitted} & \vdots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^{n-1}y_{n-1}, & D^{n-3}y_{n-1}, \ldots, & y_{n-1} \end{array}\right| = \frac{dR_m}{dx}.$$

Consider the expression

$$\frac{dR_m}{dx} Q - R_m \frac{dQ}{dx} \ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}.$$

In this the coefficient of $D^{n-r}y$ is

$$B_{r-1} = Q\Delta_{m,r} - \Delta'_{m,r} \frac{dQ}{dx} \ldots\ldots\ldots\ldots \text{(II)},$$

where $\Delta_{m,r}$ is the minor of $D^{n-r}y$ in $\frac{dR_m}{dx}$,

$\Delta'_{m,r}$ ..... R_m.

Take the series of equations (2) omitting the m^{th}, write them in the form

$$A_2D^{n-3}y_1 + A_3D^{n-4}y_1 + \ldots + A_{n-1}y_1 = -A_0D^{n-1}y_1 - A_1D^{n-2}y_1,$$
$$A_2D^{n-3}y_2 + A_3D^{n-4}y_2 + \ldots + A_{n-1}y_2 = -A_0D^{n-1}y_2 - A_1D^{n-2}y_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then solving, we have

$$A_{r-1}\Delta_{m,1} = \pm (A_0\Delta_{m,r} + A_1\Delta'_{m,r}).$$

Further, writing the same m equations (2) in the form

$$A_0D^{n-1}y_\mu + A_2D^{n-3}y_\mu + \ldots + A_{n-1}y_\mu = -A_1D^{n-2}y_\mu,$$

and solving, we have

$$A_0\frac{dQ}{dx} = -A_1Q;$$

therefore
$$\frac{-A_0}{Q} = \frac{A_1}{\frac{dQ}{dx}} = V \text{ say.}$$

Thus
$$A_{r,1}\Delta_{m,1} = \pm V\left(Q\Delta_{m,r} - \Delta'_{m,r}\frac{dQ}{dx}\right)$$
$$= \pm VB_{r-1}.$$

Moreover,
$$A_0 = \text{coefficient of } \frac{d^{n-1}y}{dx^{n-1}} \text{ in (1)}$$
$$= Q$$

and therefore
$$V = -1.$$

Hence the coefficients in (I) are proportional to those in (1); and therefore

$$\frac{dR_m}{dx}Q - R_m\frac{dQ}{dx} = \text{left-hand side of (1)}$$

multiplied by a factor which, from the comparison of the coefficients of $\frac{d^{n-1}y}{dx^{n-1}}$, is at once seen to be $\Delta_{m,1}$, that is

$$\left|\begin{array}{ccccc} D^{n-3}y_1 & , & D^{n-4}y_1, & \ldots, & y_1 \\ \ldots\ldots & & \ldots\ldots & & \ldots\ldots \\ \vdots & & y_m \text{ omitted} & & \vdots \\ \ldots\ldots & & \ldots\ldots & & \ldots\ldots \\ D^{n-3}y_{n-1}, & & D^{n-4}y_{n-1}, & \ldots, & y_{n-1} \end{array}\right| \quad \ldots\ldots\ldots\text{(III)}'.$$

Thus $$\frac{dR_m}{dx}Q - R_m\frac{dQ}{dx} = T\Delta_{m,1}$$
$$= Ae^{-\int P_1 dx}\Delta_{m,1}.$$

Hence
$$\frac{R_m}{Q} = C' + A\int\frac{e^{-\int P_1 dx}\Delta_{m,1}}{Q^2}dx,$$

or
$$R_m = C'Q + AQ\int\frac{e^{-\int P_1 dx}\Delta_{m,1}}{Q^2}dx \quad \ldots\ldots\ldots\ldots \text{(IV)}.$$

Now m may have any one of the values $1, 2, \ldots, n-1$, thus we have, in all, $n-1$ equations to give the $n-1$ quantities $y, \frac{dy}{dx}, \ldots, \frac{d^{n-2}y}{dx^{n-2}}$ which occur in the R's.

It is easy to see that the coefficient of y in R_m is $(-1)^m$ times the coefficient of y_m in Q; while that of $\frac{d^{n-r}y}{dx^{n-r}}$ in R_m is $(-1)^m$ times that of $\frac{d^{n-r}y_m}{dx^{n-r}}$ in Q; thus
$$\sum_{m=1}^{m=n-1}(-1)^m y_m R_m = yQ$$

Multiply the equation (IV) by $(-1)^m y_m$; write
$$(-1)^m C' = C_m,$$
and add for all the values of m; then the factor Q will occur on each side; and on division we obtain
$$y = \sum_{m=1}^{m=n-1}(C_m y_m) + A\sum_{m=1}^{m=n-1}\left\{(-1)^m y_m\int\frac{e^{-\int P_1 dx}\Delta_{m,1}\,dx}{Q^2}\right\},$$
where the values of $\Delta_{m,1}$ and Q are defined by the equations III and obviously are functions of the particular solutions supposed given. This expression for y contains n arbitrary constants, the $(n-1)$ C's and A; and therefore it is the general solution of the proposed equation.

A particular equation of the third order.

16. We proceed now to consider a particular equation of the third order. By substituting
$$w = ye^{-\frac{1}{3}\int P dx}$$
in the equation
$$\frac{d^3w}{dx^3} + P\frac{d^2w}{dx^2} + Q\frac{dw}{dx} + Rw = 0,$$

the equation becomes of the form

$$\frac{d^3y}{dx^3} + q\frac{dy}{dx} + ry = 0.$$

Let $z = x^{m+1}$ and $y = vz^n$ where m and n are constants yet to be determined. Substitute these values; in order that the term involving $\frac{d^2v}{dz^2}$ may be absent we shall find that

$$3m(m+1)^2 z^m x^{2m+1} + 3n(m+1)^3 x^{3m} z^{n-1} = 0,$$

and this is satisfied by

$$m + n(m+1) = 0,$$

so that

$$n = -\frac{m}{m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}.$$

Again, in order that the term in $\frac{dv}{dz}$ may be absent, we must have

$$3n(n-1)z^{n-2}(m+1)^3 x^{3m} + 6mnz^{n-1}(m+1)^2 x^{2m-1} + m(m^2-1)z^n x^{m-2} + q(m+1)x^m z^n = 0,$$

or, substituting for n, z, we have

$$-qx^2 = (m+1)^2 - 1 \quad \ldots\ldots\ldots\ldots\ldots \text{(II)}.$$

The coefficient of v

$$= n(n-1)(n-2)z^{n-3}(m+1)^3 x^{3m} + n(n-1)z^{n-2}\,3m(m+1)^2 x^{2m-1} + m(m^2-1)nz^{n-1}x^{m-2} + qnz^{n-1}(m+1)x^m + rz^n,$$

and by the use of (I) and (II), this is easily reduced to the form

$$\frac{r}{x^m} - \frac{m^2+2m}{x^{m+3}}.$$

The coefficient of $\frac{d^3v}{dz^3} = (m+1)^3 z^n x^{3m}$

$$= (m+1)^3 x^{2m}.$$

Hence the differential equation for v is

$$(m+1)^3 x^{2m}\frac{d^3v}{dz^3} + v\left[\frac{r}{x^m} - \frac{m^2+2m}{x^{m+3}}\right] = 0.$$

Let $$r = \frac{m^2 + 2m}{x^3} + Ax^p (m+1)^3 \ldots\ldots\ldots\ldots\text{(III)},$$

then v satisfies the equation

$$\frac{d^3v}{dz^3} + v \frac{A}{z^{\frac{3m-p}{m+1}}} = 0.$$

Let $p = \frac{3}{2}(m-1)$; then the equation becomes

$$z^{\frac{3}{2}} \frac{d^3v}{dz^3} + Av = 0,$$

or, without loss of generality,

$$z^{\frac{3}{2}} \frac{d^3v}{dz^3} \mp v = 0.$$

This equation can be integrated by means of Bessel's Functions (cf. Lommel, "Integration der Gleichung

$$x^{m+\frac{1}{2}} \frac{\partial^{2m+1} y}{\partial x^{2m+1}} \mp y = 0,"$$

Math. Ann. II. p. 624). If i, θ, ϕ be the three roots of

$$t^3 = -i, \quad i = \sqrt{(-1)},$$

then the general solution of

$$z^{\frac{3}{2}} \frac{d^3v}{dz^3} - v = 0$$

is
$$\begin{aligned} vz^{\frac{4}{3}} = \quad & C\left[J^{-\frac{3}{2}}(2i\sqrt{z}) + iJ^{\frac{3}{2}}(2i\sqrt{z})\right] \\ & + B\left[J^{-\frac{3}{2}}(2\theta\sqrt{z}) + iJ^{\frac{3}{2}}(2\theta\sqrt{z})\right] \\ & + A\left[J^{-\frac{3}{2}}(2\phi\sqrt{z}) + iJ^{\frac{3}{2}}(2\phi\sqrt{z})\right] \ldots\ldots\text{(IV)}. \end{aligned}$$

Obviously

$$\theta = \tfrac{1}{2}(-i + \sqrt{3}), \quad \phi = \tfrac{1}{2}(-i - \sqrt{3}).$$

Moreover, if $-i$, θ', ϕ' be the three roots of

$$t^3 = +i,$$

then the general solution of

$$z^{\frac{3}{2}} \frac{d^3v}{dz^3} + v = 0$$

is
$$\begin{aligned} vz^{\frac{4}{3}} = \; & C\left[J^{-\frac{3}{2}}(-2i\sqrt{z}) + iJ^{\frac{3}{2}}(-2i\sqrt{z})\right] \\ & + B\left[J^{-\frac{3}{2}}(2\theta'\sqrt{z}) + iJ^{\frac{3}{2}}(2\theta'\sqrt{z})\right] \\ & + A\left[J^{-\frac{3}{2}}(2\phi'\sqrt{z}) + iJ^{\frac{3}{2}}(2\phi'\sqrt{z})\right] \ldots\ldots\ldots(\mathrm{V}). \end{aligned}$$

Write $m+1=\alpha$; we have therefore

$$q = \frac{1-\alpha^2}{x^2},$$

$$r = \frac{\alpha^2-1}{x^3} + A\alpha^3 x^{-3+\frac{3\alpha}{2}},$$

Then the equation

$$\frac{d^3y}{dx^3} + \frac{1-\alpha^2}{x^2}\frac{dy}{dx} + y\left[A\alpha^3 x^{-3+\frac{3\alpha}{2}} - \frac{1-\alpha^2}{x^3}\right] = 0 \ldots(\mathrm{VI})$$

is, by the substitutions

$$\zeta = x^\alpha \quad \text{and} \quad y = v\zeta^{\frac{1-\alpha}{\alpha}},$$

transformed into

$$\zeta^{\frac{3}{2}}\frac{d^3v}{d\zeta^3} + Av = 0.$$

According as A is positive or negative, put it equal to $\mu^{\frac{3}{2}}$ or $-\mu^{\frac{3}{2}}$; then, if $z=\mu\zeta$, the two forms of the equation are respectively

$$z^{\frac{3}{2}}\frac{d^3v}{dz^3} + v = 0,$$

$$z^{\frac{3}{2}}\frac{d^3v}{dz^3} - v = 0.$$

Of the former of these (V) is the solution; of the latter, (IV); and substituting in these

$$z = A^{\frac{1}{3}}x^\alpha,$$

and writing
$$y = vx^{1-\alpha},$$

we obtain the general solution of (VI).

The expressions multiplied by the arbitrary constants in (IV) and (V) are obviously particular integrals of the differential equations to which they respectively belong; and a relation giving any one in terms of the other two might, by what has been already proved, be written down, but it would require considerable simplification before proving of use.

It is known that the ordinary hypergeometric series cannot be made to represent the Bessel's equation; it might therefore be expected (and in fact it is not difficult to prove) that the differential equation for the higher hypergeometric series cannot be made to represent the foregoing equation.

Application of Kummer's method to the general linear cubic.

17. Returning now to the differential equation

$$\frac{d^3y}{dx^3} + p\frac{d^2y}{dx^2} + q\frac{dy}{dx} + ry = 0 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

Let $y = y_1 e^{-\frac{1}{3}\int p dx}$; then the equation becomes

$$\frac{d^3y_1}{dx^3} + Q\frac{dy_1}{dx} + Ry_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

where

$$\left.\begin{aligned} Q &= q - \tfrac{1}{3}p^2 - \frac{dp}{dx} \\ R &= \tfrac{2}{27}p^3 - \tfrac{1}{3}\frac{d^2p}{dx^2} - \tfrac{1}{3}qp + r \end{aligned}\right\} \quad \ldots\ldots\ldots(3).$$

Write

$$y_1 = w_1 v_1 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

w_1 being a function of x; let v_1 be given by the equation

$$\frac{d^3v_1}{dz^3} + Q'\frac{dv_1}{dz} + R'v_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(5),$$

where z is a function of x, and Q' and R' are functions of z. It is required to find under what conditions the equations (2), (4), (5) can co-exist. Differentiating (4) we have

$$\frac{dy_1}{dx} = w_1\frac{dv_1}{dz}\frac{dz}{dx} + v_1\frac{dw_1}{dx},$$

$$\frac{d^2y_1}{dx^2} = w_1\frac{d^2v_1}{dz^2}\left(\frac{dz}{dx}\right)^2 + w_1\frac{dv_1}{dz}\frac{d^2z}{dx^2} + 2\frac{dv_1}{dz}\frac{dz}{dx}\frac{dw_1}{dx} + v_1\frac{d^2w_1}{dx^2}$$

$$\frac{d^3y_1}{dx^3} = w_1\frac{d^3v_1}{dz^3}\left(\frac{dz}{dx}\right)^3 + 3w_1\frac{d^2v_1}{dz^2}\frac{dz}{dx}\frac{d^2z}{dx^2} + 3\frac{d^2v}{dz^2}\left(\frac{dz}{dx}\right)^2\frac{dw_1}{dx}$$

$$+ w_1\frac{dv_1}{dz}\frac{d^3z}{dx^3} + 3\frac{dw_1}{dx}\frac{dv_1}{dz}\frac{d^2z}{dx^2} + 3\frac{dv_1}{dz}\frac{dz}{dx}\frac{d^2w_1}{dx^2} + v_1\frac{d^3w_1}{dx^3}.$$

Substituting these in (2) the resulting equation is linear in v_1, and must therefore be the same as (5); hence comparing the coefficients, we have

$$0 = 3w_1 \frac{dz}{dx}\frac{d^2z}{dx^2} + 3\left(\frac{dz}{dx}\right)^2 \frac{dw_1}{dx},$$

$$w_1\left(\frac{dz}{dx}\right)^3 Q' = w_1\frac{d^3z}{dx^3} + 3\frac{dw_1}{dx}\frac{d^2z}{dx^2} + 3\frac{d^2w_1}{dx^2}\frac{dz}{dx} + Qw_1\frac{dz}{dx} \ldots (6),$$

$$w_1\left(\frac{dz}{dx}\right)^3 R' = \frac{d^3w_1}{dx^3} + Q\frac{dw_1}{dx} + Rw_1 \ldots\ldots\ldots\ldots (7),$$

and from the first of these

$$w_1\frac{dz}{dx} = A, \text{ a constant} \ldots\ldots\ldots\ldots (8).$$

Can these three equations be satisfied by the same values of w_1 and z?

Denoting $\dfrac{dz}{dx}$ by z' we have, on the elimination of w_1 between (6) and (8),

$$Q'z'^2 = -2\frac{z'''}{z'} + 3\left(\frac{z''}{z'}\right)^2 + Q,$$

or, adopting Cayley's notation (*Camb. Phil. Trans.* t. XIII., 'On the Polyhedral Functions')

$$\frac{z'''}{z'} - \tfrac{3}{2}\left(\frac{z''}{z'}\right)^2 = \{z, x\},$$

the above is

$$\{z, x\} + \tfrac{1}{2}Q'\left(\frac{dz}{dx}\right)^2 - \tfrac{1}{2}Q = 0 \ldots\ldots\ldots\ldots (9),$$

an equation identical in form with that obtained by Kummer (*Crelle*, t. XV. p. 46).

Eliminating w_1 between (7) and (8) we have

$$\frac{z''''}{z'} - 6\frac{z'''z''}{z'^2} + 6\frac{z''^3}{z'^3} + Q\frac{z''}{z'} - R = -R'z'^3;$$

i.e., $$\frac{d}{dx}\{z, x\} - 2\frac{z''}{z'}\{z, x\} + Q\frac{z''}{z'} - R = -R'z'^3 \ldots (10).$$

The two equations (9) and (10) must be satisfied by the same value of z. Now from (9)

$$\frac{d}{dx}\{z, x\} = \tfrac{1}{2}\frac{dQ}{dx} - \tfrac{1}{2}\frac{dQ'}{dz}z'^3 - Q'z''z',$$

and from (9), (10)

$$\frac{d}{dx}\{z, x\} = R - R'z'^3 - Q'z''z';$$

therefore
$$\tfrac{1}{2}\frac{dQ}{dx} - R = \left\{\tfrac{1}{2}\frac{dQ'}{dz} - R'\right\}\left(\frac{dz}{dx}\right)^3 \text{.....(11)},$$

and we may consider (9) and (11) as the equations to be satisfied by the value of z.

If we have obtained (5) from

$$\frac{d^3v}{dz^3} + p'\frac{d^2v}{dz^2} + q'\frac{dv}{dz} + r' = 0 \text{ ..... (1)'},$$

by writing $v_1^{-\frac{1}{3}}e^{\int p'dz} = v$, then Q' and R' will be the same function of p', q', and r' as those which occur in (3) are of p, q and r. Suppose

$$y = wv;$$

then
$$w = w_1 e^{-\frac{1}{3}\int p dx + \frac{1}{3}\int p' dz},$$

and therefore
$$w\frac{dz}{dx} = Ae^{-\frac{1}{3}(\int p dx - \int p' dz)} \text{ .....(8)'},$$

an equation which will give the value of w when a suitable value of z, should any exist, has been found.

Deduction of the particular integrals of the differential equation to the hypergeometric series.

18. Applying the foregoing to the particular case of the hypergeometric series of the order 2, then we find

$$\tfrac{1}{2}Q = \frac{Ax^2 + Bx + C}{x^2(1-x)^2},$$

so that (9) is

$$\{z, x\} + \frac{A'x^2 + B'z + C'}{z^2(1-z)^2}\left(\frac{dz}{dx}\right)^2 - \frac{Ax^2 + Bx + C}{x^2(1-x)^2} = 0,$$

where A, B, C are functions of the elements α, β, γ, θ, ε and A', B', C' of α', β', θ', γ', ε'.

If with Kummer (*l. c.*) we consider z to be a function of x alone and not to depend on these elements, then there are six values of z suitable, viz.:—

$$(1)\ z=x,\quad (2)\ z=\frac{1}{x},\quad (3)\ z=1-x,\quad (4)\ z=\frac{1}{1-x},$$

$$(5)\ z=\frac{x}{x-1},\quad (6)\ z=\frac{x-1}{x},$$

and for each of these

$$\{z, x\}=0.$$

Taking the first of these the conditions to be satisfied are obviously

$$Q'=Q \text{ and } R'=R,$$

or as R is of the form

$$\frac{Dx^3+Ex^2+Fx+G}{x^3(1-x)^3}$$

the conditions are that each of the functions A, B, C, D, E, F, G of α, β, θ, γ, ε shall be equal to the corresponding functions of α', β', θ', γ', ε' respectively.

For shortness, write

$$\mathrm{a}=\alpha\beta+\beta\theta+\theta\alpha+\alpha+\beta+\theta+1,\quad \mathrm{b}=\alpha+\beta+\theta+3,$$

$$\mathrm{c}=\gamma\varepsilon,\quad \mathrm{d}=\gamma+\varepsilon+1,\quad \mathrm{e}=\alpha\beta\theta;$$

then

$$2A=\ \mathrm{a}-\tfrac{1}{3}\mathrm{b}^2+\mathrm{b},\qquad 2B=\mathrm{a}+\mathrm{c}-\tfrac{2}{3}\mathrm{bd}+2\mathrm{d},$$

$$2C=\ \mathrm{c}-\tfrac{1}{3}\mathrm{d}^2+\mathrm{d},\qquad G=\frac{2\mathrm{d}^3}{27}-\tfrac{2}{3}\mathrm{d}-\tfrac{1}{3}\mathrm{cd},$$

$$D=-\tfrac{2}{27}\mathrm{b}^3+\tfrac{2}{3}\mathrm{b}+\tfrac{1}{3}\mathrm{ab}-\mathrm{e},$$

$$E=\ \tfrac{2}{3}\mathrm{db}^2-2\mathrm{d}-\tfrac{1}{3}(\mathrm{cb}+\mathrm{ba}+\mathrm{ad})+2\mathrm{e},$$

$$F=-\tfrac{2}{3}\mathrm{d}^2\mathrm{b}+2\mathrm{d}+\tfrac{1}{3}(\mathrm{ad}+\mathrm{dc}+\mathrm{cb})-\mathrm{e}.$$

There will be only five of these quantities independent; the other two must be expressible in terms of them. Let the five independent quantities be taken to be A, B, C, G, D; then our conditions will be satisfied if

$$A=A',\quad B=B',\quad C=C',\quad D=D',\quad G=G'.$$

It will be noticed that C, G involve γ and ε alone; in fact

$$6C = \gamma\varepsilon + \gamma + \varepsilon + 2 - \gamma^3 - \varepsilon^3,$$

$$27G = 2\gamma^3 + 2\varepsilon^3 - 3\gamma^2\varepsilon - 3\varepsilon^2\gamma + 6\gamma^2 + 6\varepsilon^2 + 3\gamma\varepsilon - 12\gamma - 12\varepsilon - 16.$$

19. We proceed to find the values of γ' and ε' which will satisfy the conditions, and shall retain only those which are linear functions of γ and ε. Hence the values will be of the form

$$\varepsilon' = a + b\varepsilon + c\gamma,$$

$$\gamma' = d + g\varepsilon + f\gamma.$$

Substituting in the relation

$$C = C',$$

and equating the coefficients of corresponding terms we obtain

$$ad - a^2 - d^2 + a + d = 0,$$

$$ag + bd - 2dg - 2ab + b + g = 1,$$

$$dc + af - 2df - 2ac + c + f = 1,$$

$$b^2 + g^2 - bg = 1 \quad \text{..... (i)},$$

$$c^2 + f^2 - cf = 1 \quad \text{..... (ii)},$$

$$bf + cg - 2fg - 2bc = 1 \quad \text{.....(iii)}.$$

Writing $a = 1 + a'$, $d = 1 + d'$, the first three of these equations become

$$a'^2 + d'^2 - a'd' = 1 \quad \text{..... (iv)},$$

$$b(d' - 2a') + g(a' - 2d') = 1 \quad \text{..... (v)},$$

$$c(d' - 2a') + f(a' - 2d') = 1 \quad \text{.....(vi)}.$$

By (v), (vi)

$$\frac{d' - 2a'}{g - f} = \frac{a' - 2d'}{c - b} = \frac{1}{cg - bf}$$

$$= \frac{-3a'}{c - b + 2(g - f)} = \frac{-3d'}{g - f + 2(c - b)} = -3T, \text{ say.}$$

Substituting in (iv) for a', d' and using (i), (ii), (iii) we find

$$T = \pm \tfrac{1}{3}.$$

Hence $$cg - bf = 1 \quad \text{.....(vii)},$$

while $$\left.\begin{aligned} a - 1 &= -\tfrac{1}{3}(c-b) - \tfrac{2}{3}(g-f) \\ d - 1 &= -\tfrac{1}{3}(g-f) - \tfrac{2}{3}(c-b) \end{aligned}\right\},$$

or $$cg - bf = -1 \quad \text{.....(vii)'},$$

while $$\left.\begin{aligned} a - 1 &= \tfrac{1}{3}(c-b) + \tfrac{2}{3}(g-f) \\ d - 1 &= \tfrac{1}{3}(g-f) + \tfrac{2}{3}(c-b) \end{aligned}\right\}.$$

Taking the equations (i), (ii), (iii), (vii) it is not difficult to prove that they are all satisfied by

$$\left.\begin{aligned} b + f &= 0 \\ g + f &= c \\ f^2 + g^2 + fg &= 1 \end{aligned}\right\} \quad \text{.....(I).}$$

(these including all the solutions) and then

$$\left.\begin{aligned} a + g &= 1 \\ c + d &= 1 \end{aligned}\right\}$$

Taking next (i), (ii), (iii) and (vii)' these are all satisfied completely by

$$\left.\begin{aligned} c + g &= 0 \\ g + f &= b \\ f^2 + g^2 + fg &= 1 \\ a + f &= 1 \\ b + d &= 1 \end{aligned}\right\} \quad \text{..... (II).}$$

20. Substituting now the values of γ' and ε' in the condition

$$G = G',$$

and equating the coefficients of corresponding terms we obtain the following equations:—

$$\begin{aligned}
2 &= 2f^3 + 2c^3 - 3cf(c+f), \\
2 &= 2g^3 + 2b^3 - 3bg(b+g), \\
1 &= -2f^2g - 2c^2b + fc(b+g) + (bf+cg)(c+f), \\
1 &= -2fg^2 - 2cb^2 + bg(c+f) + (bf+cg)(b+g), \\
2 &= 2f^2d + 2c^2a + 2f^2 + 2c^2 + fc(1-a-d) - (c+f)(cd+af), \\
2 &= 2g^2d + 2b^2a + 2g^2 + 2b^2 + bg(1-a-d) - (b+g)(bd+ag), \\
1 &= 4fgd + 4cba + 4fg + 4bc + (1-a-d)(bf+cg) \\
&\qquad - (b+g)(cd+af) - (c+f)(bd+ag),
\end{aligned}$$

$$-4 = 2fd^2 + 2ca^2 + 4fd + 4ca - 4f - 4c - ad(c+f) + (1-a-d)(cd+af),$$

$$-4 = 2gd^2 + 2ba^2 + 4gd + 4ba - 4g - 4b - ad(b+g) + (1-a-d)(bd+ag),$$

$$0 = 2a^3 + 2d^3 + 3ad(1-a-d) + 6a^2 + 6d^2 - 12a - 12d.$$

Combining set I with the second of the above equations and solving we find that the following values are the only ones obtainable, and they satisfy all the remaining equations:—

$$f=0,\ g=1,\ b=0,\ c=1,\ a=0,\ d=0 \ldots (A),$$

$$f=-1,\ g=0,\ b=1,\ c=-1,\ a=1,\ d=2 \ldots (B),$$

$$f=1,\ g=-1,\ b=-1,\ c=0,\ a=2,\ d=1 \ldots (C).$$

Similarly treat set II with the first of the above: only three solutions are obtained and they satisfy all the equations; they are

$$f=0,\ g=-1,\ b=-1,\ c=1,\ a=1,\ d=2 \ldots (D),$$

$$f=1,\ g=0,\ b=1,\ c=0,\ a=0,\ d=0 \ldots (E),$$

$$f=-1,\ g=1,\ b=0,\ c=-1,\ a=2,\ d=1 \ldots (F).$$

Now

$$(A) \text{ gives } \left.\begin{matrix}\gamma' = \varepsilon \\ \varepsilon' = \gamma\end{matrix}\right\}, \qquad (E) \text{ gives } \left.\begin{matrix}\gamma' = \gamma \\ \varepsilon' = \varepsilon\end{matrix}\right\},$$

$$(B) \ldots\ldots \left.\begin{matrix}\gamma' = 2-\gamma \\ \varepsilon' = 1-\gamma+\varepsilon\end{matrix}\right\}, \qquad (F) \ldots\ldots \left.\begin{matrix}\gamma' = 1-\gamma+\varepsilon \\ \varepsilon' = 2-\gamma\end{matrix}\right\},$$

$$(C) \ldots\ldots \left.\begin{matrix}\gamma' = 1-\varepsilon+\gamma \\ \varepsilon' = 2-\varepsilon\end{matrix}\right\}, \qquad (D) \ldots\ldots \left.\begin{matrix}\gamma' = 2-\varepsilon \\ \varepsilon' = 1-\varepsilon+\gamma\end{matrix}\right\}.$$

These are obviously, so far as the hypergeometric series are concerned, identical in the horizontal pairs.

21. We proceed now to solve for α', β', θ'; and for this purpose we have the relations

$$A = A', \quad B = B', \quad D = D'.$$

In the second (and the second alone) of these γ and ε enter, and they enter symmetrically in the forms $\gamma\varepsilon$ and $\gamma+\varepsilon$, so that the above pairs are equivalent, as they ought to be; and three cases, corresponding to the three pairs, will arise.

We have

$$-12A = (\alpha-\beta)^2+(\beta-\theta)^2+(\theta-\alpha)^2-6,$$

$$6B = 3(\alpha\beta+\beta\theta+\theta\alpha+\gamma\varepsilon)-2(\alpha+\beta+\theta)(\gamma+\varepsilon)+\alpha+\beta+\theta+3,$$

$$-27D = 2(\alpha+\beta+\theta+3)^3+27\alpha\beta\theta$$
$$-9(\alpha+\beta+\theta+3)(\alpha\beta+\beta\theta+\theta\alpha+\alpha+\beta+\theta+3).$$

Retaining those values of the desired elements $\alpha'\beta'\theta'$ which are linear in terms of the others, let

$$\alpha' = a\alpha+b\beta+c\theta+d\varepsilon+f\gamma+g,$$
$$\beta' = h\alpha+j\beta+k\theta+d\varepsilon+f\gamma+g,$$
$$\theta' = l\alpha+m\beta+n\theta+d\varepsilon+f\gamma+g,$$

the parts independent of α, β, θ being assumed the same, since we are to have

$$A = A'.$$

Substitute and equate coefficients; then writing

$$a-h=L, \quad b-j=M, \quad c-k=N,$$
$$h-l=A, \quad j-m=B, \quad k-n=C,$$

it will be found (after some merely algebraical work) that the only solutions of the equations are given by

(1) $A=N, \quad C=M, \quad M^2+MN+N^2=1,$
$$-B=-L=M+N.$$

(2) $A=M, \quad B=N, \quad M^2+MN+N^2=1,$
$$-C=-L=M+N.$$

22. *Case I.* $\gamma'\varepsilon'=\gamma\varepsilon$ and $\gamma'+\varepsilon'=\gamma+\varepsilon$.

Using the condition $B=B'$, substituting the values of α', β' and θ' on the right-hand side and equating coefficients, we obtain a series of twenty-one equations; and these, after considerable reductions, are equivalent to

$$ah+hl+la=0,$$
$$bj+jm+mb=0,$$
$$ck+kn+nc=0,$$
$$ab+hj+lm=0,$$
$$bc+jk+mn=0,$$
$$ca+kh+nl=0,$$

and one of the two sets

$$\begin{cases} d = f = 0 = g, \\ a+h+l = b+j+m = c+k+n = 1, \end{cases}$$

$$\begin{cases} d = f = \frac{2}{3}, \quad g = -\frac{1}{3}, \\ a+h+l = b+j+m = c+k+n = -1. \end{cases}$$

Let $a+h+l = \mu$, so that $\mu = \pm 1$.

Then, using the relation $D = D'$ and proceeding as before, we obtain a series of equations to be satisfied by the quantities. Among these we find

$$2\mu^3 + 27ahl = 2.$$

First, let $\mu = 1$; then

$$ahl = 0,$$

from which, with the other relations, we obtain the result that two of the quantities a, h, l are zero and the remaining one unity. Similarly for b, j, m and for c, k, n.

The other equations are not satisfied unless one only out of each of the sets a, b, c; h, j, k; l, m, n; be not zero, and thus we have

$$\left.\begin{matrix} \alpha' = \alpha \\ \beta' = \beta \\ \theta' = \theta \end{matrix}\right\}, \qquad \left.\begin{matrix} \gamma' = \gamma \\ \varepsilon' = \varepsilon \end{matrix}\right\},$$

the quantities bracketed being permutable.

Second, let $\mu = -1$; then

$$27ahl = 4.$$

Thus a, h, l are the roots of the equation

$$\xi^3 + \xi^2 - \tfrac{4}{27} = 0;$$

that is, are $\quad \frac{1}{3}, \quad -\frac{2}{3}, \quad -\frac{2}{3}.$

The same results hold with regard to b, j, m and to c, k, n; and all the equations are satisfied provided

$$\left.\begin{matrix} 3\alpha' = \quad \alpha - 2\beta - 2\theta + 2\varepsilon + 2\gamma - 1 \\ 3\beta' = -2\alpha + \beta - 2\theta + 2\varepsilon + 2\gamma - 1 \\ 3\theta' = -2\alpha - 2\beta + \theta + 2\varepsilon + 2\gamma - 1 \end{matrix}\right\}, \qquad \left.\begin{matrix} \gamma' = \gamma \\ \varepsilon' = \varepsilon \end{matrix}\right\},$$

the bracketed quantities being permutable.

23. *Case II.* $\gamma'\varepsilon' = (2-\gamma)(1-\gamma+\varepsilon)$, $\gamma'+\varepsilon' = 3-2\gamma+\varepsilon$.

One solution we already know to be

$$\left.\begin{aligned}\alpha' &= \alpha+1-\gamma\\ \beta' &= \beta+1-\gamma\\ \theta' &= \theta+1-\gamma\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= 2-\gamma\\ \varepsilon' &= 1-\gamma+\varepsilon\end{aligned}\right\},$$

and from the foregoing we infer that there is only one other, given by

$$\left.\begin{aligned}3\alpha' &= \alpha-2\beta-2\theta-\gamma+2\varepsilon+2\\ 3\beta' &= -2\alpha+\beta-2\theta-\gamma+2\varepsilon+2\\ 3\theta' &= -2\alpha-2\beta+\theta-\gamma+2\varepsilon+2\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= 2-\gamma\\ \varepsilon' &= 1-\gamma+\varepsilon\end{aligned}\right\}.$$

24. *Case III.* $\gamma'\varepsilon' = (2-\varepsilon)(1+\gamma-\varepsilon)$, $\gamma'+\varepsilon' = 3-2\varepsilon+\gamma$.

The two solutions in this case are

$$\left.\begin{aligned}\alpha' &= \alpha+1-\varepsilon\\ \beta' &= \beta+1-\varepsilon\\ \theta' &= \theta+1-\varepsilon\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= 1-\varepsilon+\gamma\\ \varepsilon' &= 2-\varepsilon\end{aligned}\right\},$$

and

$$\left.\begin{aligned}3\alpha' &= \alpha-2\beta-2\theta-\varepsilon+2\gamma+2\\ 3\beta' &= -2\alpha+\beta-2\theta-\varepsilon+2\gamma+2\\ 3\theta' &= -2\alpha-2\beta+\theta-\varepsilon+2\gamma+2\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= 1-\varepsilon+\gamma\\ \varepsilon' &= 2-\varepsilon\end{aligned}\right\}.$$

The multipliers w in the six solutions in order are

$$w = c,$$
$$w = c(1-x)^{\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1},$$
$$w = cx^{1-\gamma},$$
$$w = cx^{1-\gamma}(1-x)^{\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1},$$
$$w = cx^{1-\varepsilon},$$
$$w = cx^{1-\varepsilon}(1-x)^{\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1}.$$

25. Take now the value $z = \dfrac{1}{x}$.

Hence, if $D' = z\dfrac{d}{dz}$ and $D = x\dfrac{d}{dx}$,

$$D' = -D.$$

Let $$y = x^{\mu} v;$$

then $$Dy = x^{\mu} (D + \mu) v,$$

and therefore $$(D + \lambda) y = x^{\mu} (- D' + \lambda + \mu) v,$$

so that the equation

$$\left[(D+\alpha)(D+\beta)(D+\theta) - \frac{1}{x} D(D+\gamma-1)(D+\varepsilon-1) \right] y = 0$$

transforms into

$$[(-D'+\alpha+\mu)(-D'+\beta+\mu)(-D'+\theta+\mu) - z(-D'+\mu)(-D'+\gamma-1+\mu)(-D'+\varepsilon-1+\mu)] v = 0,$$

or, writing $$\mu = -\alpha,$$

$$\left[(D'+\alpha)(D'+\alpha+1-\gamma)(D'+\alpha+1-\varepsilon) - \frac{1}{z} D'(D'+\alpha-\beta)(D'+\alpha-\theta) \right] v = 0,$$

from which it is obvious that

$$v = F\left\{ \begin{pmatrix} \alpha, & \alpha+1-\gamma, & \alpha+1-\varepsilon \\ \alpha-\beta+1, & \alpha-\theta+1 & \end{pmatrix}, z \right\},$$

or one solution of the original equation is given by

$$\left.\begin{aligned} \alpha' &= \alpha \\ \beta' &= \alpha+1-\gamma \\ \theta' &= \alpha+1-\theta \end{aligned}\right\}, \qquad \left.\begin{aligned} \gamma' &= \alpha-\beta+1 \\ \varepsilon' &= \alpha-\theta+1 \end{aligned}\right\},$$

and the factor w is $Cx^{-\alpha}$.

26. The remaining five solutions belonging to this case can now be written down from a comparison with the last. They are:

$$\left.\begin{aligned} 3\alpha' &= \alpha - 2\beta - 2\theta + 2\varepsilon + 2\gamma - 1 \\ 3\beta' &= -2\alpha + \beta - 2\theta + 2\varepsilon + 2\gamma - 1 \\ 3\theta' &= -2\alpha - 2\beta + \theta + 2\varepsilon + 2\gamma - 1 \end{aligned}\right\}, \qquad \left.\begin{aligned} \gamma' &= \alpha-\beta+1 \\ \varepsilon' &= \alpha-\theta+1 \end{aligned}\right\},$$

and the factor w is

$$Cx^{-\alpha+\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1} (1-x)^{\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1};$$

$$\left.\begin{aligned}\alpha' &= \beta + 1 - \gamma\\ \beta' &= \beta\\ \theta' &= \beta + 1 - \varepsilon\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= \beta - \alpha + 1\\ \varepsilon' &= \beta - \theta + 1\end{aligned}\right\},$$

and the factor w is $Cx^{-\beta}$;

$$\left.\begin{aligned}3\alpha' &= \alpha - 2\beta - 2\theta + 2\varepsilon + 2\gamma - 1\\ 3\beta' &= -2\alpha + \beta - 2\theta + 2\varepsilon + 2\gamma - 1\\ 3\theta' &= -2\alpha - 2\beta + \theta + 2\varepsilon + 2\gamma - 1\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= \beta - \alpha + 1\\ \varepsilon' &= \beta - \theta + 1\end{aligned}\right\},$$

and the factor w is

$$Cx^{-\beta+\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1}(1-x)^{\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1};$$

$$\left.\begin{aligned}\alpha' &= \theta + 1 - \gamma\\ \beta' &= \theta + 1 - \varepsilon\\ \theta' &= \theta\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= \theta - \alpha + 1\\ \varepsilon' &= \theta - \beta + 1\end{aligned}\right\},$$

and the factor w is $Cx^{-\theta}$; while, lastly,

$$\left.\begin{aligned}3\alpha' &= \alpha - 2\beta - 2\theta + 2\varepsilon + 2\gamma - 1\\ 3\beta' &= -2\alpha + \beta - 2\theta + 2\varepsilon + 2\gamma - 1\\ 3\theta' &= -2\alpha - 2\beta + \theta + 2\varepsilon + 2\gamma - 1\end{aligned}\right\}, \qquad \left.\begin{aligned}\gamma' &= \theta - \alpha + 1\\ \varepsilon' &= \theta - \beta + 1\end{aligned}\right\},$$

and the factor w is

$$Cx^{-\theta+\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1}(1-x)^{\frac{1}{3}(2\gamma+2\varepsilon-2\alpha-2\beta-2\theta)-1}.$$

27. Consider now the value $z = 1 - x$.

In this case $\dfrac{dz}{dx} = -1$, and therefore

$$Ax^2 + Bx + C = A'z^2 + B'z + C',$$

and therefore $A = A'$, $2A + B = B'$,

$$A + B + C = C'.$$

Take the last of these as involving only γ' and ε' of the new elements and write

$$\gamma' = p\alpha + q\beta + r\theta + s\varepsilon + t\gamma + u,$$
$$\varepsilon' = e\alpha + v\beta + w\theta + x\varepsilon + y\gamma + z.$$

Substitute and equate the terms which are independent of all combinations of the old elements, and we have

$$uz + u + z + 2 - u^2 - z^2 = 8,$$

i.e. $$(u-1)^2 + (z-1)^2 + (u-z)^2 = -10,$$

an impossible equation. There are therefore no solutions of the form required, in the case of the substitution $z = 1 - x$.

28. From the form of Kummer's three remaining substitutions which are all derived from a combination of $z=\frac{1}{x}$ and $z=1-x$ it follows that there are no sets of values of $\alpha', \beta', \theta', \gamma', \varepsilon'$ suitable to the conditions laid down; and we have therefore twelve particular integrals of the differential equation. Let

$$-(2\alpha+2\beta+2\theta)+2\gamma+2\varepsilon=3\mu+1.$$

Then these integrals are

(i) $$F\left\{\begin{pmatrix}\alpha, & \beta, & \theta\\ \gamma, & \varepsilon & \end{pmatrix}, x\right\},$$

(ii) $$(1-x)^{\mu}F\left\{\begin{pmatrix}\alpha+\mu, & \beta+\mu, & \theta+\mu\\ \gamma, & \varepsilon & \end{pmatrix}, x\right\},$$

(iii) $x^{1-\gamma}$ $$F\left\{\begin{pmatrix}\alpha+1-\gamma, & \beta+1-\gamma, & \theta+1-\gamma\\ 2-\gamma, & \varepsilon+1-\gamma & \end{pmatrix}, x\right\},$$

(iv) $x^{1-\gamma}$ $$(1-x)^{\mu}F\left\{\begin{pmatrix}\alpha+\mu+1-\gamma, & \beta+\mu+1-\gamma, & \theta+\mu+1-\gamma\\ 2-\gamma, & \varepsilon+1-\gamma & \end{pmatrix}, x\right\},$$

(v) $x^{1-\varepsilon}$ $$F\left\{\begin{pmatrix}\alpha+1-\varepsilon, & \beta+1-\varepsilon, & \theta+1-\varepsilon\\ \gamma+1-\varepsilon, & 2-\varepsilon & \end{pmatrix}, x\right\},$$

(vi) $x^{1-\varepsilon}$ $$(1-x)^{\mu}F\left\{\begin{pmatrix}\alpha+\mu+1-\varepsilon, & \beta+\mu+1-\varepsilon, & \theta+\mu+1-\varepsilon\\ \gamma+1-\varepsilon, & 2-\varepsilon & \end{pmatrix}, x\right\},$$

(vii) $x^{-\alpha}$ $$F\left\{\begin{pmatrix}\alpha, & \alpha+1-\gamma, & \alpha+1-\varepsilon\\ \alpha-\beta+1, & \alpha-\theta+1 & \end{pmatrix}, \frac{1}{x}\right\},$$

(viii) $x^{-(\alpha+\mu)}(1-x)^{\mu}F\left\{\begin{pmatrix}\alpha+\mu, & \beta+\mu, & \theta+\mu\\ \alpha-\beta+1, & \alpha+\theta+1 & \end{pmatrix}, \frac{1}{x}\right\},$

(ix) $x^{-\beta}$ $$F\left\{\begin{pmatrix}\beta+1-\gamma, & \beta, & \beta+1-\varepsilon\\ \beta-\alpha+1, & \beta-\theta+1 & \end{pmatrix}, \frac{1}{x}\right\},$$

(x) $x^{-(\beta+\mu)}(1-x)^{\mu}F\left\{\begin{pmatrix}\alpha+\mu, & \beta+\mu, & \theta+\mu\\ \beta-\alpha+1, & \beta-\theta+1 & \end{pmatrix}, \frac{1}{x}\right\},$

(xi) $x^{-\theta}$ $$F\left\{\begin{pmatrix}\theta+1-\gamma, & \theta+1-\varepsilon, & \theta\\ \theta-\alpha+1, & \theta-\beta+1 & \end{pmatrix}, \frac{1}{x}\right\},$$

(xii) $x^{-(\theta+\mu)}(1-x)^{\mu}F\left\{\begin{pmatrix}\alpha+\mu, & \beta+\mu, & \theta+\mu\\ \theta-\alpha+1, & \theta-\beta+1 & \end{pmatrix}, \frac{1}{x}\right\}.$

Let these be denoted respectively by

$$\psi_{\mathrm{i}}, \psi_{\mathrm{ii}}, \psi_{\mathrm{iii}}, \psi_{\mathrm{iv}}, \ldots, \psi_{\mathrm{xi}}, \psi_{\mathrm{xii}}.$$

Then among these there subsist the relations

$$\psi_{\mathrm{i}} = \psi_{\mathrm{ii}}, \qquad \psi_{\mathrm{iii}} = \psi_{\mathrm{iv}}, \qquad \psi_{\mathrm{v}} = \psi_{\mathrm{vi}},$$

$$\psi_{\mathrm{vii}} = (-)^{\mu}\psi_{\mathrm{viii}}, \qquad \psi_{\mathrm{ix}} = (-)^{\mu}\psi_{\mathrm{x}}, \qquad \psi_{\mathrm{xi}} = (-)^{\mu}\psi_{\mathrm{xii}}.$$

If x be real, then for values numerically less than unity the first six particular integrals are converging series while the last six diverge; and for values of x numerically greater than unity the last six converge while the first six diverge. This is of course on the assumption that the series are convergent for the value unity of x; the condition to be satisfied in order to ensure this convergency is in the case of ψ_{i} that $\gamma + \varepsilon > \alpha + \beta + \theta$; and then ψ_{iii}, ψ_{v}, ψ_{vii}, ψ_{ix}, ψ_{xi} are convergent. Again ψ_{ii}, ψ_{iv}, ψ_{vi} are convergent if $\gamma + \varepsilon < \alpha + \beta + \theta + 1$; while for the convergency of ψ_{viii} $3\alpha + 3 > 2\gamma + 2\varepsilon$, for that of ψ_{x} $3\beta + 3 > 2\gamma + 2\varepsilon$, and for that of ψ_{xii} $3\theta + 3 > 2\gamma + 2\varepsilon$.

Differential equation satisfied by the quotient of two particular integrals of the linear cubic.

29. Of the questions analogous to those treated in the theory of the hypergeometric series of the first order the only remaining one which I propose to investigate is the deduction of the differential equation which is satisfied by the quotient of two particular solutions of the equation representing the hypergeometric series of the second order. Since this quotient is of the form

$$\frac{A\psi_{\mathrm{i}} + B\psi_{\mathrm{iii}} + C\psi_{\mathrm{v}}}{A'\psi_{\mathrm{i}} + B'\psi_{\mathrm{iii}} + C'\psi_{\mathrm{v}}},$$

the differential equation required will obviously be of the fifth order.

Let the second term—involving $\frac{d^2y}{dx^2}$—be removed from the differential equation of the series, so that it becomes of the form

$$\frac{d^3y}{dx^3} + Q\frac{dy}{dx} + Ry = 0,$$

and let y_1 and y_2 be two particular solutions, so that

$$\frac{d^3y_1}{dx^3} + Q\frac{dy_1}{dx} + Ry_1 = 0,$$

$$\frac{d^3y_2}{dx^3} + Q\frac{dy_2}{dx} + Ry_2 = 0.$$

Let $$V = y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx};$$

then $$\frac{dV}{dx} = y_2\frac{d^2y_1}{dx^2} - y_1\frac{d^2y_2}{dx^2},$$

$$\frac{d^2V}{dx^2} = y_2\frac{d^3y_1}{dx^3} - y_1\frac{d^3y_2}{dx^3} + \frac{1}{y_2}\frac{dy_2}{dx}\frac{dV}{dx} - \frac{V}{y_2}\frac{d^2y_2}{dx} \ldots (A),$$

so that $$\frac{d^2V}{dx^2} - \frac{1}{y_2}\frac{dy_2}{dx}\frac{dV}{dx} + \frac{V}{y_2}\frac{d^2y_2}{dx^2} = -QV.$$

Let $s = \frac{y_1}{y_2}$; then (dashes denoting differentials of s with regard to x)

$$s' = \frac{V}{y_2^2},$$

$$\frac{s''}{s'} = \frac{1}{V}\frac{dV}{dx} - 2\frac{1}{y_2}\frac{dy_2}{dx},$$

$$\frac{s'''}{s'} - \left(\frac{s''}{s'}\right)^2 = \frac{1}{V}\frac{d^2V}{dx^2} - \frac{1}{V^2}\left(\frac{dV}{dx}\right)^2 - \frac{2}{y_2}\frac{d^2y}{dx^2} + 2\left(\frac{1}{y_2}\frac{dy_2}{dx}\right)^2$$

$$-\tfrac{1}{2}\left(\frac{s''}{s'}\right)^2 = -\tfrac{1}{2}\frac{1}{V^2}\left(\frac{dV}{dx}\right)^2 + 2\frac{1}{y_2}\frac{dy_2}{dx}\frac{1}{V}\frac{dV}{dx} - 2\left(\frac{1}{y_2}\frac{dy_2}{dx}\right)^2.$$

For brevity let

$$\frac{s'''}{s'} - \tfrac{3}{2}\left(\frac{s''}{s'}\right)^2 = \{s, x\} = \sigma;$$

then

$$\sigma = \frac{1}{V}\frac{d^2V}{dx^2} - \tfrac{3}{2}\left(\frac{1}{V}\frac{dV}{dx}\right)^2 + \frac{2}{y_2}\frac{dy_2}{dx}\frac{1}{V}\frac{dV}{dx} - \frac{2}{y_2}\frac{d^2y_2}{dx^2} \ldots (B)$$

$$= \frac{3}{V}\frac{d^2V}{dx^2} - \tfrac{3}{2}\left(\frac{1}{V}\frac{dV}{dx}\right)^2 + 2Q \ldots\ldots\ldots\ldots\ldots\ldots (B').$$

It is convenient to reduce the order of this equation; write

$$\frac{1}{V}\frac{dV}{dx} = 2W \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

and the equation becomes

$$\frac{dW}{dx} + W^2 = \frac{\sigma - 2Q}{6} \ldots\ldots\ldots\ldots\ldots\ldots (D).$$

Differentiating (B) we obtain

$$\begin{aligned}\frac{d\sigma}{dx} &= \frac{1}{V}\frac{d^3V}{dx^3} - \frac{4}{V^2}\frac{dV}{dx}\frac{d^2V}{dx^2} + \frac{3}{V^3}\left(\frac{dV}{dx}\right)^3 \\ &\quad + 2\frac{1}{y_2}\frac{dy_2}{dx}\left[\frac{1}{V}\frac{d^2V}{dx^2} - \frac{1}{V^2}\left(\frac{dV}{dx}\right)^2 - \frac{1}{y_2}\frac{dy_2}{dx}\frac{1}{V}\frac{dV}{dx} + \frac{1}{y_2}\frac{d^2y_2}{dx^2}\right] \\ &\quad + \frac{2}{y_2}\frac{d^2y_2}{dx^2}\frac{1}{V}\frac{dV}{dx} - \frac{2}{y_2}\frac{d^3y_2}{dx^3} \\ &= \frac{1}{V}\frac{d^3V}{dx^3} - \frac{6}{V^2}\frac{dV}{dx}\frac{d^2V}{dx^2} + \frac{3}{V^3}\left(\frac{dV}{dx}\right)^3 + 2R - \frac{2Q}{V}\frac{dV}{dx},\end{aligned}$$

after a slight reduction obtained by means of (B), (B') and the equation satisfied by y_2.

But also by differentiating (B')

$$\frac{d\sigma}{dx} = \frac{3}{V}\frac{d^3V}{dx^3} - \frac{6}{V^2}\frac{dV}{dx}\frac{d^2V}{dx^2} + \frac{3}{V^3}\left(\frac{dV}{dx}\right)^3 + 2\frac{dQ}{dx}.$$

Multiplying the former by 3 and subtracting the latter we have

$$\begin{aligned}2\frac{d\sigma}{dx} &= -\frac{4}{V}\frac{dV}{dx}\left[\frac{3}{V}\frac{d^2V}{dx^2} - \frac{3}{2V^2}\left(\frac{dV}{dx}\right)^2 + \tfrac{3}{2}Q\right] + 6R - 2\frac{dQ}{dx} \\ &= -8W\left[\sigma - \tfrac{1}{2}Q\right] + 6R - 2\frac{dQ}{dx},\end{aligned}$$

or

$$W\left[\sigma - \tfrac{1}{2}Q\right] = \tfrac{1}{4}\left[3R - \frac{dQ}{dx} - \frac{d\sigma}{dx}\right] \ldots\ldots\ldots (E).$$

Eliminating W between (D) and (E) we have a differential equation of the second order in σ, and one therefore of the fifth order in s. This agrees with the result obtained by Prof. Malet (*Phil. Trans.* 1882, p. 759, "On a class of Invariants").*

---

* In the function T (l. c.) there is a mistake; it should be

$$T = G - \tfrac{3}{2}\frac{f'''^3}{f'^3} + 2\frac{f''f'''}{f''^2} - \frac{f''''}{2f'}.$$

As a special example we may put $Q=0$, $R=0$, in which case

$$s=\frac{A+Bx+Cx^2}{A'+B'x+C'x^2}$$

(the quotient of two solutions of $\frac{d^3y}{dx^3}=0$); and the differential equation satisfied by s is

$$\frac{d^2\sigma}{dx^2}-\frac{5}{4\sigma}\left(\frac{d\sigma}{dx}\right)^2+\tfrac{2}{3}\sigma^2=0,$$

where, it will be remembered,

$$\sigma=\{s,\,x\}=\frac{s'''}{s'}-\tfrac{3}{2}\left(\frac{s''}{s'}\right)^2.$$

NOTE.—In addition to the authorities already quoted, there are two papers on the generalised form of the hypergeometric series which may be consulted. They are:

POCKHAMMER—"Ueber hypergeometrische Functionen n^{ter} Ordnung," *Crelle*, t. 71 (1870), pp. 316–352.

THOMÆ—"Ueber die höheren hypergeometrischen Reihen, insbesondere über die Reihe

$$1+\frac{a_0.a_1.a_2}{1.b_1.b_2}x+\frac{a_0(a_0+1)\,a_1(a_1+1)\,a_2(a_2+1)}{1.2.b_1(b_1+1)\,b_2(b_2+1)}x^2+\text{etc.},$$

Math. Ann. II. (1870), pp. 427–444.

ON PROFESSOR CAYLEY'S CANONICAL FORM.

By Captain P. A. MACMAHON, R.A.

§1. PROF. CAYLEY has recently shown in the *Quart. Math. Journ.*, Feb. 1883, that his canonical form for the binary quintic belongs to one of a series. I purpose here to show the law of formation of the canonical coefficients in terms of the fundamental seminvariants, and to deduce therefrom some theorems relating to seminvariants generally, and to the theory of the irreducible covariants of binary quantics.

The source of the canonical form is

$$\frac{1}{\mathrm{a}}(\mathrm{a},\,\mathrm{b},\,\mathrm{c},\,\mathrm{d},\,\ldots\chi x-\mathrm{b}y,\,\mathrm{a}y)^n$$

for the n^{ic}; or, putting $a = y = 1$ for convenience,

$$(1, b, c, d, \ldots)(x - b, 1)^n \ldots\ldots\ldots\ldots\ldots (1),$$

and this is written

$$(1, 0, C, D, \ldots)(x, 1)^n,$$

where

$$C = c - b^2,$$
$$D = d - 3bc + 2b^3,$$
$$E = e - 4bd + 6b^2c - 3b^4,$$
$$F = f - 5be + 10b^2d - 10b^3c + 4b^5,$$
$$\&c. \qquad \&c.$$

Now it is to be observed that the canonical quantic (1) has its roots simply each less by b than the quantic

$$(1, b, c, d, \ldots)(x, 1)^n \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

since it is obtained from the latter by the substitution $x - b$ for x; consequently, since seminvariants are functions of the differences of the roots, the seminvariants of the canonical quantic (1) must be identical with those of the quantic (2); following Prof. Cayley, and representing the fundamental seminvariants of the quantic (2) (viz. those of alternately two and three degrees respectively) by the small italic letters

$$a, b, c, d, e, \ldots,$$

and, for convenience, putting $a = 1$, and at the same time forming the fundamental seminvariants of the canonical quantic

$$(1, 0, C, D, \ldots)(x, 1)^n,$$

we arrive at the following system of equations, viz.

$$1 = 1,$$
$$C = c = c - b^2,$$
$$D = d = d - 3bc + 2b^3,$$
$$E + 3C^2 = e = e - 4bd + 3c^2,$$
$$F + 2CD = f = f - 5be + 2cd + 8b^2d - 6bc^2,$$
$$G + 15CE - 10D^2 = g = g - 6bf + 15ce - 10d^2,$$
$$H + 9CF - 5DE = h = h - 7bg + 9cf - 5de + 12b^2f - 30bce + 20bd^2,$$
$$I + 28CG - 56DF + 35E^2 = i = i - 8bh + 28cg - 56df + 35e^2,$$
$$\&c. \qquad \&c.,$$

so that the capital forms are simply the small roman ones with b = 0, and the letters changed into capitals.

Hence we derive at once:

$$E = e - 3c^2,$$

$$F = f - 2cd,$$

$$G = g - 15c\,(e - 3c^2) + 10d^2$$
$$= g - 15ce + 10d^2 + 45c^3,$$

$$H = h - 9c\,(f - 2cd) + 5d\,(e - 3c^2)$$
$$= h - 9cf + 5de + 3c^2d,$$

$$I = i - 28c(g - 15be + 10d^2 + 45c^3) + 56d\,(f - 2cd) - 35\,(e - 3c^2)^2$$
$$= i - 28cg + 56df - 35e^2 + 630c^2e - 392cd^2 - 1575c^4,$$

results which agree with those already calculated by Prof. Cayley.

It will be seen that the number of terms in the expression of the weight w coefficient is the number of non-unitary partitions of w. We obtain in like manner the values of J, K, L, M, and, presenting all the results in a tabular form, we have

$E =$	$F =$	$G =$	$H =$	$I =$	$J =$	$K =$	$L =$	$M =$
$e + 1$	$f + 1$	$g + 1$	$h + 1$	$i + 1$	$j + 1$	$k + 1$	$l + 1$	$m + 1$
$c^2 - 3$	$cd - 2$	$d^2 + 10$	$de + 5$	$dh - 20$	$dg + 28$	$di - 45$	$dj + 75$	$dk - 66$
		$c^3 + 45$	$c^2d + 3$	$cg - 28$	$ch - 20$	$dh + 120$	$di - 75$	$dj + 220$
				$df + 56$	$df - 14$	$eg - 210$	$eh - 90$	$ei - 495$
				$e^2 - 35$	$cf + 222$	$f^2 + 126$	$fg + 42$	$fh + 792$
				$c^2e + 630$	$ce - 492$	$cf + 1890$	$c^2h + 970$	$g^2 - 462$
				$cd^2 - 392$	$d^3 + 280$	$cdf - 4104$	$cdg - 3164$	$c^2i + 4455$
				$c^4 - 1575$	$c^3d + 1116$	$ce^2 + 4725$	$cef + 670$	$cdh - 13904$
						$d^2e - 1500$	$d^2f + 4620$	$ceg + 41580$
						$c^3e - 47250$	$de^2 - 3075$	$d^2g + 6160$
						$c^2d^2 + 24804$	$c^3f - 8310$	$ef^2 - 15444$
						$c^5 + 99225$	$c^2de + 66810$	$def - 26840$
							$cd^3 - 40040$	$e^3 + 17325$
							$c^4d - 160155$	$c^3g - 20790$
								$c^2df + 41949$
								$c^2e^2 - 77962$
								$cd^2e + 31548$
								$d^4 + 15400$
								$c^4e + 48336$
								$c^3d^2 - 23942$
								$c^6 - 98232$

§ 2. From the above it is clear that if ϕ (c, d, e, ...) be that portion of a seminvariant, the partitions of whose terms are non-unitary, then the complete expression of the seminvariant is

$$\phi\,(C, D, E, \ldots),$$

the capital letters being the canonical coefficients; *ex. gr.* given

$$\phi(c, d, e, \ldots) = ce - d^2 - c^3,$$

$$\phi(C, D, E) = (c-b^2)(e-4bd+6b^2c-3b^4)-(d-3bc+2b^3)^2-(c-b^2)^3$$
$$= ce - d^2 - b^2e + 2bcd - c^3,$$

which, of course, is the invariant T of the quartic, and so on in every case a seminvariant is completely given by its non-unitary portion.

I have shown elsewhere (*vide Amer. Journ. of Math.*) that every seminvariant is expressible in terms of non-unitary symmetric functions, and now we find that it is only necessary to know the non-unitary terms of a seminvariant to obtain its complete expression; the theory is thus brought to depend merely on non-unitary partitions both of symmetric functions and of products and powers of coefficients.

Hence we arrive immediately at the point of being able to express any given seminvariant at once in terms of the fundamental forms, for we have seen above how thus to express the canonical coefficients.

This theory involves therefore that of the syzygies, and, in the case of a given system of seminvariants, contains potentially its complete solution.

As an example, consider the seminvariant which, in Mr. Hammond's notation, is HR, viz.

$$2c^2h - 7cdg + 5cef - 4b^2ch + 7b^2dg + 7bc^2g - 5b^2ef + 22bcdf$$
$$- 27c^3f - 25bce^2 + 45c^2de - 20cd^3 + 2b^4h - 7b^3cg - 22b^3df$$
$$+ 27b^2c^2f + 25b^3e^2 - 45b^2cde + 20b^2d^3.$$

Here

$$\phi(c, d, e, f, g, h) = 2c^2h - 7cdg + 5cef - 27c^3f + 45c^2de - 20cd^3$$

The seminvariant is therefore

$$\phi(C, D, E, F, G, H)$$
$$= 2C^2H - 7CDG + 5CEF - 27C^3F + 45C^2DE - 20CD^3,$$

or

$$HR = 2c^2(h - 9cf + 5de + 3c^2d) - 7cd(g - 15ce + 10d^2 + 45c^3)$$
$$+ 5c(e - 3c^2)(f - 2cd) - 27c^3(f - 2cd) + 45c^2d(e - 3c^2) - 20cd^3$$
$$= c\{2ch - 7dg + 5ef - 30c(2cf - 5de) - 90d(d^2 + 4c^3)\},$$

which agrees with Mr. Hammond's result (*vide Amer. Journ.*, vol. v. No. 3, p. 223).

The above example of a composite seminvariant has been chosen, because it demonstrates that any canonical-coefficient-seminvariant factor, which a form may contain, is at once discovered from a mere inspection of the non-unitary portion; thus, c being a factor of the non-unitary portion, we saw at once that C and therefore c must be a factor of the complete seminvariant.

Generally if $f(\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \ldots)$ be a factor of the non-unitary portion of a seminvariant, then $f(C, D, E, F, \ldots)$; that is,

$$f(c, d, e - 3c^2, f - 2cd, \ldots)$$

must be a factor of the whole seminvariant.

This theorem is very useful in the theory of syzygies.

§ 3. If we regard the small italic letters as the fundamental seminvariants of the quantic

$$(1, 0, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \ldots)(x, 1)^n,$$

then this latter quantic may be written

$$(1, 0, c, d, e - 3c^2, f - 2cd, \ldots)(x, 1)^n,$$

which is identical in expression with the canonical form, and in no simpler way can a quantic be written, so that its coefficients may be functions of its own fundamental seminvariants.

It may be also noted that, considering the canonical form of the cubic, viz.

$$x^3 + 3cx + d,$$

and considering therein c, d as denoting not the sources but the covariants themselves, so that c is the Hessian H, and d is the cubic covariant J, then the form is

$$x^3 + 3Hx + J,$$

which has been shown by Prof. Cayley to be the form assumed by the cubic when the Tschirnhausen transformation, is applied to it (*vide* Salmon's *Higher Algebra*, third edition p. 300).

THE "TRIPLICATE-RATIO" CIRCLE.

By R. Tucker, M.A.

LET α, β, γ be the coordinates of a point (P) taken within the triangle ABC, so that if parallels EPF' (to BC), DPE' to (AB), and FPD' (to CA) be drawn, the six points D, D'; E, E'; F, F' shall lie on a circumference (fig. 19).

We have the following equalities:

$$\left.\begin{array}{l} BF' = \alpha \operatorname{cosec} B \\ BD = \gamma \operatorname{cosec} B \end{array}\right\}, \quad \left.\begin{array}{l} AE = \beta \operatorname{cosec} A \\ AE' = \gamma \operatorname{cosec} A \end{array}\right\}, \quad \left.\begin{array}{l} CE = \alpha \operatorname{cosec} C \\ CD' = \beta \operatorname{cosec} C \end{array}\right\} \ldots\text{(i)}.$$

Also the following:

$$\frac{BF}{BD'} = \frac{BA}{BC} = \frac{c}{a}, \quad \frac{AF'}{AE} = \frac{c}{b}, \quad \frac{CE'}{CD} = \frac{b}{a},$$

whence $$\frac{BF}{c} = \frac{BD'}{a}, \quad \frac{AF'}{c} = \frac{AE}{b}, \quad \frac{CE'}{b} = \frac{CD}{a} \ldots\ldots\ldots\text{(ii)}.$$

By the hypothesis,

$$\left.\begin{array}{l} BD'.BD = a.p.\gamma \operatorname{cosec} B \\ BF'.BF = c.p.\alpha \operatorname{cosec} B \end{array}\right\},$$

whence $$\frac{\alpha}{a} = \frac{\gamma}{c} = \frac{\beta}{b} = \frac{2\Delta}{a^2+b^2+c^2} = \frac{2\Delta}{K} \ldots\ldots\ldots\text{(iii)}.$$

Let AP, BP, CP produced meet the sides in K, L, M respectively, then

$$\frac{a}{b} = \frac{\alpha}{\beta} = \frac{\sin PCB}{\sin PCA} = \frac{BM}{a} \cdot \frac{b}{AM};$$

therefore $$\frac{BM}{AM} = \frac{a^2}{b^2}, \quad \frac{AL}{CL} = \frac{c^2}{a^2}, \quad \frac{CK}{BK} = \frac{b^2}{c^2} \ldots\ldots\ldots\text{(iv)}.$$

We get also the equalities

$$\left.\begin{array}{lll} BD = \dfrac{2c\Delta}{K \sin B} = \dfrac{ac^2}{K}, & BF' = \dfrac{a^2c}{K}, & AF = \dfrac{b^2c}{K} \\ AE' = \dfrac{bc^2}{K}, & CE = \dfrac{a^2b}{K}, & CD' = \dfrac{ab^2}{K} \end{array}\right\} \ldots\text{(v)},$$

$$\left.\begin{array}{lll} BF = \dfrac{c}{K}(a^2+c^2), & AF' = \dfrac{c}{K}(b^2+c^2), & CE' = \dfrac{b}{K}(a^2+b^2) \\ BD' = \dfrac{a}{K}(a^2+c^2), & AE = \dfrac{b}{K}(b^2+c^2), & CD = \dfrac{a}{K}(a^2+b^2) \end{array}\right\} \ldots\text{(vi)}.$$

Also

$$K^2.FD'^2 = a^2c^4 + c^2(a^2+c^2)^2 - 2ac^2.c(a^2+c^2)\cos B$$
$$= c^2(a^2b^2 + b^2c^2 + c^2a^2) = c^2\lambda^2;$$

therefore $FD = \frac{c\lambda}{K} = c\mu, \quad EF = b\mu, \quad DE = a\mu.$

Similarly $F'D' = a\mu, \quad E'F' = c\mu, \quad D'E' = b\mu.$ } ...(vii).

Hence the triangles DEF, $D'E'F'$ are similar to ABC and equal to one another, having

$$\angle A = DFE = D'E'F', \quad \angle B = EDF = E'F'D',$$
$$\angle C = E'D'F' = DEF.$$

Again,

$$K^2.DF'^2 = a^2c^4 + a^4c^2 - 2a^3c^3\cos B = a^2b^2c^2$$

hence $$DF' = \frac{abc}{K} = ED' = FE' \quad \ldots\ldots\ldots\ldots \text{(viii)},$$

and

$$D'F = \frac{b(a^2+c^2)}{K}, \quad E'D = \frac{c(a^2+b^2)}{K}, \quad F'E = \frac{a(b^2+c^2)}{K} \ldots\text{(ix)};$$

also $$DD' = \frac{a^3}{K}, \quad EE' = \frac{b^3}{K}, \quad FF' = \frac{c^3}{K}* \ldots\ldots\ldots\text{(x)}.$$

If DE' cuts $D'F'$, EF in g, g'; EF' cuts $D'E'$, DF in h, h'; and FD' cuts $E'F'$, DE in k, k', then

$$\frac{F'k}{E'k} = \frac{F'P}{EP} = \frac{BK}{CK} = \frac{c^2}{b^2}, \quad \frac{D'g}{F'g} = \frac{a^2}{c^2}, \quad \frac{E'h}{D'h} = \frac{b^2}{a^2} \ldots\text{(xi)}.$$

Similarly,

$$\frac{Dk'}{Ek'} = \frac{a^2}{b^2}, \quad \frac{Fh'}{Dh'} = \frac{c^2}{a^2}, \quad \frac{Eg'}{Fg'} = \frac{b^2}{c^2} \ldots\ldots\ldots\text{(xii)}.$$

From (viii) we have

$$\angle BFD = \angle FDE' = \angle D'FE = \&c. = \omega,$$

then $$\cos\omega = \frac{c^2(a^2+c^2)^2 + c^2\lambda^2 - a^2c^4}{2c^2(a^2+c^2)\lambda} = \frac{K}{2\lambda} \ldots\ldots\ldots\text{(xiii)},$$

$$\sin^2\omega = \frac{4\lambda^2 - K^2}{4\lambda^2} = \frac{2a^2b^2 + \ldots - a^4 - b^4 - c^4}{4\lambda^2} = \frac{16\Delta^2}{4\lambda^2},$$

---

* It is this property which has suggested the title of "Triplicate-Ratio" Circle. It is obvious to remark that the sides are cut in duplicate ratio by the lines AP, BP, CP produced, and are also cut by the circle into three segments which are in the duplicate ratio of the sides.

therefore $$\sin\omega = \frac{2\Delta}{\lambda} \quad \text{.....(xiv)};$$

hence $$\sin 2\omega = \frac{2\Delta K}{\lambda^2},$$

and $$\left.\begin{aligned}\cot\omega = \frac{K}{4\Delta} &= \frac{a^2+b^2+c^2}{4\Delta} = \frac{bc\cos A + \ldots + \ldots}{bc\sin A} \\ &= \cot A + \cot B + \cot C\end{aligned}\right\} \ldots \text{(xv)},$$

We can also readily prove that, if

$$\angle AFE = \omega_1 \quad \text{and} \quad \angle AE'F'' = \omega'_1,$$

then $$\omega_1 = \omega'_1,$$

and similarly for the other angles; therefore

$$\left.\begin{aligned}\omega_1 = \omega_1' &= B + C - \omega \\ \omega_2 = \omega_2' &= C + A - \omega \\ \omega_3 = \omega_3' &= A + B - \omega\end{aligned}\right\},$$

whence $$\omega_1 + \omega_2 + \omega_3 = \omega'_1 + \omega'_2 + \omega'_3 = 2\pi - 3\omega.$$

Also, if $$\Delta_1 = \Delta AFE, \quad \Delta' = \Delta AF'E',$$

then $$\Delta_1 + \Delta_2 + \Delta_3 = \frac{\Delta\,(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)}{K^2}$$

$$= \Delta'_1 + \Delta'_2 + \Delta'_3,$$

and $$\Delta EPF = E'PF = E'PF' = AFE' \ldots\ldots\ldots \text{(xvi)},$$

and so on.

By (viii) and (x) we get

$$\text{perimeter of } DD'EE'FF' = \frac{a^3+b^3+c^3+3abc}{K}.$$

$$\text{and its area} = \frac{\Delta}{k^2}\,(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)$$

$$= \Delta - \Delta' = \Delta\,(1-\mu^2);$$

$$\text{but area also} = \Delta - (AFE' + BDF' + CED');$$

therefore $$\Delta AFE' + BDF' + CED' = \Delta',$$

where Δ' is ΔDEF (or $\Delta D'E'F'$).

Let p_1, &c., p'_1, &c., be the perpendiculars from P on the sides of DEF, $D'E'F'$, then

$$p_1 = PE \sin\omega = \frac{ab^2}{K} \cdot \frac{2\Delta}{\lambda}, \qquad p'_1 = \frac{ac^2 \sin\omega}{K};$$

therefore

$$\left.\begin{aligned} p_1 p'_1 &= \frac{a^2b^2c^2 \sin^2\omega}{K^2} = p_2 p'_2 = p_3 p'_3 = 4\rho^2 \sin^4\omega \\ \text{and} \quad p_1 p_2 p_3 &= \frac{a^3b^3c^3 \sin^3\omega}{K^3} = \quad p'_1 p'_2 p'_3 \quad = 8\rho^3 \sin^6\omega \end{aligned}\right\} \ldots\text{(xvii)}.$$

Since

$$PF.PD' = AE'.CE = \frac{bc^2}{K} \cdot \frac{a^2b}{K} = \frac{a^2b^2c^2}{K^2} = DF'^2,$$

therefore minimum chord through $P = 2DF'$.

By (vii) $\mu R = \rho$, where ρ is the radius of the triplicate-ratio circle.

By (xiii) and (xiv), since

$$\cos DFD' = \cos(A - \omega) = \cos A \frac{K}{2\lambda} + \sin A \frac{2\Delta}{\lambda} = \frac{bc + a^2 \cos A}{\lambda},$$

therefore

$$\text{ordinate of centre} = \rho \cos DFD' = \frac{abc}{4\Delta} \cdot \frac{bc + a^2 \cos A}{K},$$

$$\text{abscissa of centre} = \frac{a(a^2 + 2c^2)}{2K},$$

and $$\text{abscissa of } P = \frac{ac(c + a\cos B)}{K};$$

therefore

$$\delta^2 = O_1P^2 = \ldots = \frac{a^2b^2c^2}{16\Delta^2K^2}(a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2)$$

$$= \frac{\rho^2}{\lambda^2}(\lambda^2 - 16\Delta^2) = \frac{\rho^2(K^2 - 3\lambda^2)}{\lambda^2} \ldots\ldots\text{(xviii)}.$$

If we denote $\angle BAK$, &c., by θ_1, &c., and $\angle CAK$, &c., by θ'_1, &c., we have

$$\frac{\sin(\theta_1 + B)}{\sin\theta_1} = \frac{b^2 + c^2}{ac};$$

therefore $$\cot\theta_1 = \frac{b^2 + bc\cos A}{2\Delta},$$

similarly $$\cot\theta'_1 = \frac{c^2 + bc\cos A}{2\Delta};$$

hence

$$\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = \frac{3K}{4\Delta} = 3\cot\omega = \cot\theta'_1 + \cot\theta'_2 + \cot\theta'_3.$$

By means of (v), or otherwise, it is easily seen that $BCE'F'$, $ACDF'$, $ABD'E$ are circumscriptible.

The trilinear equation to the triplicate-ratio circle can be put into the following shapes:

$$abc(\alpha^2 + \beta^2 + \gamma^2) = \frac{abc}{K}(a\alpha + b\beta + c\gamma)^2 + a^3\beta\gamma + b^3\gamma\alpha + c^3\alpha\beta$$

and .....(xix),

$$(b^2 + c^2)\alpha^2 + (c^2 + a^2)\beta^2 + (a^2 + b^2)\gamma^2$$

$$= \{(a^2 + b^2)(a^2 + c^2) + b^2c^2\}\frac{\beta\gamma}{bc} + \{(b^2 + c^2)(b^2 + a^2) + c^2a^2\}\frac{\gamma\alpha}{ca}$$

$$+ \{(c^2 + a^2)(c^3 + b^2) + a^2b^2\}\frac{\alpha\beta}{ab}.$$

I had got so far when Mr. Glaisher put at my disposal his copy of the "Etude d'un nouveau cercle du plan du triangle," par M. H. Brocard (*Association Française pour l'Avancement des Sciences—Congrès d'Alger*, 1881). On first looking at the results given in this pamphlet I was much surprised to find some of the results given above, but was glad to see that the two circles were distinct. A reference to fig. 19 will show how it is the circles are so intimately connected. M. Brocard's circle has the following properties: through A, B, C draw straight lines parallel to the sides of the two triangles DEF, $D'E'F'$, these intersect in two points O, O', which have been called *Brocard* points, and these lines intersect again in three points (A_1, B_1, C_1 not figured), then the *Brocard* circle is that circle on whose circumference the seven points O, O', P, A_1, B_1, C_1, and H (centre of circle ABC) lie.

I get its trilinear equation in the form

$$abc(\alpha^2 + \beta^2 + \gamma^2)$$

$$= \frac{16\Delta^2}{K}(a\beta\gamma + b\gamma\alpha + c\alpha\beta) + a^3\beta\gamma + b^3\gamma\alpha + c^3\alpha\beta \ldots(xx);$$

this, by (xix), shows that the two circles are concentric, as can be shown geometrically.

The equation (xx) can be put into the simpler form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + \frac{1}{K}(\alpha+\beta+\gamma)(a\alpha+b\beta+c\gamma) = 0 \ldots (\text{xxi}).$$

It is easily verified that this circle passes through P and H. If ρ' be the radius of the "Brocard" circle, then (p. 7 *l.c.*)

$$\rho' = 2OO' \frac{\sin 2\omega}{1 - 3\tan^2\omega} \ldots\ldots\ldots\ldots (\text{xxii});$$

this will be found to agree with the result of (xviii).

We have

$$BO = \frac{a\sin(C-\omega)}{\sin C} = \frac{ac^2}{\lambda};$$

hence coordinates of O are

$$\left.\begin{aligned} &\alpha_1 = \frac{ac^2\sin\omega}{\lambda}, \quad \beta_1 = \frac{bc^2\sin\omega}{\lambda}, \quad \gamma_1 = \frac{ab^2\sin\omega}{\lambda}. \\ &\text{Similarly, for } O', \text{ we get} \\ &\alpha'_1 = \frac{ab^2\sin\omega}{\lambda}, \quad \beta'_1 = \frac{bc^2\sin\omega}{\lambda}, \quad \gamma'_1 = \frac{ca^2\sin\omega}{\lambda}, \end{aligned}\right\} \ldots (\text{xxiii}),$$

whence $$\alpha_1\alpha'_1 = \frac{a^2b^2c^2\sin^2\omega}{\lambda^2} = \beta_1\beta'_1 = \gamma_1\gamma'_1 \ldots\ldots\ldots (\text{xxiv}).$$

The equation to HOP is

$$\alpha\sin(B-C) + \beta\sin(C-A) + \gamma\sin(A-B) = 0 \ldots (\text{xxv}),$$

and to OO' is

$$\frac{\alpha}{a}(a^4 - b^2c^2) + \frac{\beta}{b}(b^4 - c^2a^2) + \frac{\gamma}{c}(c^4 - a^2b^2) = 0 \ldots\ldots (\text{xxvi}).$$

I have found the following results bearing upon the "T. R." circle. In the *Reprint* from the *Educational Times* (vol. IX. p. 79) Mr. S. Roberts finds the equation to the *ellipse* through the points $D, D', \ldots F'$: when the coordinates of P are substituted for his point O I get a circle whose equation is that given in (xix); and in the *Reprint* (vol. XXII. p. 58) Prof. Wolstenholme has a question on an inscribed ellipse, and a comparison of results obtained in his and my solutions

(p. 60) with results obtained in this paper will suggest many further properties.*

From Question V. (with its solution) in the *Lady's and Gentleman's Diary* for 1859 (1933) it appears that if perpendiculars are let fall on the sides from O, O', the two triangles formed by joining their feet are the minimum inscribed triangles "such that their sides respectively make the same angle (θ) with the sides of the given triangle" are equal to each other and have the same circumscribing circle, whose centre is the mid-point of OO'. In this question $\theta = \frac{1}{2}\pi - \omega$.

[The result at foot of p. 344 readily comes thus:

$$\Delta\, AFE' = AFP = EPF \text{ and so on.}$$

I have received elegant geometrical solutions of some of the above properties, which have been set as Question 7414 in the *Educational Times* for July, 1883, from Miss C. A. Scott. She proposes to call O (got by the cyclic order $OAB = OBC = OCA$) the first Brocard point, O' the second point, and ω the Brocard angle. From (xi) and (xii) it follows that P is the second Brocard point for DEF and the first Brocard point for $D'E'F'$. The points A_1, B_1, C_1 can be most readily found by letting fall perpendiculars from H on DE', &c.

M. Neuberg has added to (xv) the useful equality

$$\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

The sides $D'E$, $E'F$, $F'D$ are clearly parallel to the sides of the pedal triangle of ABC, and, if T be the orthocentre, it may be shown at once that a circle, centre O_1, and radius equal $\frac{1}{2}R$, touches $D'E$, $E'F$, $F'D$ at their mid-points, and that the line joining the centres of the "T. R." circle and the nine-point circle is parallel to, and equal half of, PT. We hence derive

$$\rho^2 = \frac{R^2}{4K^2}\,(16\Delta^2 + K^2).]$$

---

* See especially results (xxiii), (xxiv). Numerous other results may be obtained, and those which are given in the paper admit of being proved in more ways than one; the method which first occurred to me is in most instances that which I have given. From (viii) and (x) we get

$$DD'.\,EE'.\,FF' = \frac{a^3b^3c^3}{K^3} = DF'.\,ED'.\,FE',\ \&c.$$

ON CERTAIN PHYSICAL PROBLEMS CONNECTED WITH SURFACES WHICH ARE THE INVERSES OF ELLIPSOIDS OF REVOLUTION.

By A. B. Basset, M.A.

1. In a paper which was published in the present volume of this *Journal*, I discussed certain hydrodynamical and electro-statical problems connected with cylinders whose transverse sections are the inverses of ellipses; the object of the present paper is to consider certain problems of a similar character connected with the surfaces formed by the revolution of these curves about their axes. In the paper alluded to the form and singularities of these curves are fully discussed, and expressions obtained for various quantities which occur in terms of conjugate functions; it will therefore be unnecessary in the present paper to enter into any detailed examination of the properties of these surfaces; I shall consequently simply quote such formulae as are required and refer to my former paper for their demonstration. It will hereafter appear that the surfaces which are the inverses of planetary ellipsoids, require a slightly different treatment from that employed in the case of the surfaces which are the inverses of ovary ellipsoids; I have therefore divided this paper into two parts, the first of which deals with the latter, whilst the second deals with the former class of surfaces.

Part I.

2. Laplace's equation when transformed into cylindrical coordinates x, ρ, θ becomes

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{d\rho^2}+\frac{1}{\rho}\frac{d\phi}{d\rho}+\frac{1}{\rho^2}\frac{d^2\phi}{d\theta^2}=0 \quad \ldots\ldots\ldots\ldots(1).$$

Let $$\phi = V\sin(m\theta+\alpha_m),$$

where V is a function of x and ρ only; substituting in (1), the equation for determining V is

$$\frac{d^2V}{dx^2}+\frac{d^2V}{d\rho^2}+\frac{1}{\rho}\frac{dV}{d\rho}-\frac{m^2V}{\rho^2}=0 \quad \ldots\ldots\ldots\ldots(2).$$

Let $$V=\frac{U}{\sqrt{\rho}},$$

and (2) becomes

$$\frac{d^2U}{dx^2}+\frac{d^2U}{d\rho^2}+\frac{1}{\rho^2}\left(\tfrac{1}{4}-m^2\right)U=0 \quad \ldots\ldots\ldots\ldots(3).$$

Let $$x + \iota\rho = f(\xi + \iota\eta),$$

$$J^2 = \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{d\rho}\right)^2 = \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{d\rho}\right)^2,$$

and (3) becomes

$$\frac{d^2U}{d\xi^2} + \frac{d^2U}{d\eta^2} + \frac{1}{J^2\rho^2}\left(\tfrac{1}{4} - m^2\right)U = 0 \ldots\ldots\ldots\ldots (4).$$

Now if $U = W\sqrt{(uv)}$ where u is a function of ξ alone, and v a function of η, (4) may be thrown into the form

$$\frac{1}{u}\frac{d}{d\xi}\left(u\frac{dW}{d\xi}\right) + \frac{1}{v}\frac{d}{d\eta}\left(v\frac{dW}{d\eta}\right)$$

$$+ \frac{1}{J^2\rho^2}\left(\tfrac{1}{4} - m^2\right)W + \left(\frac{u''}{2u} - \frac{u'^2}{4u^2} + \frac{v''}{2v} - \frac{u'^2}{4v^2}\right)W = 0 \ldots (5),$$

the accents denoting differentiation. Hence if $\dfrac{1}{J^2\rho^2}$ is a function of either ξ or η only, or if it is the sum of two such functions, we can express W in a series, each term of which is of the form $X_n Y_n$, where X_n is a function of ξ, and Y_n is a function of η only.

3. For the purpose of dealing with problems connected with the surfaces under consideration, we must put

$$x + \iota\rho = c\sec(\xi + \iota\eta),$$

the axis of x being the axis of revolution; hence

$$\frac{1}{J^2\rho^2} = \operatorname{cosec}^2\xi + \operatorname{cosech}^2\eta.$$

Also let $$u = \sin\xi, \quad v = \sinh\eta,$$

$$\mu = \cos\xi, \quad \nu = \cosh\eta.$$

Making these substitutions, (5) becomes

$$\frac{d}{du}(1-\mu^2)\frac{dW}{d\mu} - \frac{d}{dv}(1-\nu^2)\frac{dW}{dv} - \left(\frac{1}{1-\mu^2} - \frac{1}{1-\nu^2}\right)m^2W = 0.$$

This equation is satisfied by the series $\Sigma X_n Y_n$, where X_n and Y_n respectively satisfy the equations

$$\frac{d}{d\mu}(1-\mu^2)\frac{dX_n}{d\mu} + \left(C - \frac{m^2}{1-\mu^2}\right)X_n = 0,$$

$$\frac{d}{dv}(1-\nu^2)\frac{dY_n}{dv} + \left(C - \frac{m^2}{1-\nu^2}\right)Y_n = 0,$$

and C is some constant, hence

$$\phi = \frac{1}{r} \Sigma X_n Y_n \Sigma \sin (m\theta + \alpha_m).$$

Since ϕ is periodic with regard to θ, m must be an integer. When the potential is symmetrical about the axis, ϕ is independent of θ and $m = 0$; it remains therefore to determine C. Now we know that the *variable* parts of potentials of the charges upon an ellipsoid, when the surface is (1) under the action of no electric forces, (2) under the action of a uniform force parallel to x, are respectively equal to

$$\int_\lambda^\infty \frac{d\psi}{\{(a^2+\psi)(b^2+\psi)(c^2+\psi)\}^{\frac{1}{2}}},$$

and

$$x \int_\lambda^\infty \frac{d\psi}{\{(a^2+\psi)^3(b^2+\psi)(c^2+\psi)\}^{\frac{1}{2}}},$$

where

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

Inverting with respect to the centre we obtain the potentials of the induced changes upon the interior of the inverse surface, when the surface encloses (1) a charge at the centre, (2) a system whose potential is proportional to $\frac{x}{r^3}$. Making $b = c$ and evaluating the definite integrals, we shall find that the variable parts of the last-mentioned potentials are respectively equal to $\frac{Q_0(\nu)}{r}$ and $\frac{P_1(\mu)\ Q_1(\nu)}{r}$, where $\sqrt{(a^2-b^2)}\,\nu = \sqrt{(a^2+\lambda)}$, and P and Q are the zonal harmonics of the first and second kinds respectively. We therefore conclude that X_n and Y_n are zonal harmonics, and therefore that $C = n(n+1)$.

Hence, when ϕ is symmetrical about the axis, its general value is

$$\frac{1}{r} \Sigma \{A_n P_n(\nu) + B_n Q_n(\nu)\}\ P_n(\mu).$$

Along the lines extending from the foci to infinity $\nu = 1$, and therefore $Q_n(\nu)$ is infinite; hence the Q functions cannot enter into the expression for ϕ in any case in which the potential is finite along these lines. On the other hand, ν and therefore $P_n(\nu)$ is infinite at the origin, so that ϕ cannot contain the P functions in any case in which the potential is finite at the origin. But $\frac{Q_n}{r}$ is finite at the origin and $\frac{P_n}{r}$

at the foci, so that the Q functions are suitable for space which includes the origin, and the P functions for space which includes the foci; but where the space considered includes neither of these regions both functions may occur.

When the potential is unsymmetrical about the axis the general value of ϕ is

$$\frac{1}{r}\Sigma\{A_n T_n^m(\nu) + B_n t_n^m(\nu)\}\, T_n^m(\mu)\, \Sigma \sin(m\theta + \alpha_n),$$

where T_n^m and t_n^m are the factors involving ν of the tesseral harmonics of the first and second kinds respectively, and are respectively equal to

$$(\nu^2-1)^{\frac{1}{2}m}\frac{d^m P_n}{d\nu^m} \text{ and } (\nu^2-1)^{\frac{1}{2}m}\frac{d^m Q_n}{d\nu^m}.$$

These functions are subject to limitations as regards their employment, similar to those which we have shown to exist in the case of the P and Q functions.

4. The functions we are now considering are too well known to require discussion, we shall therefore simply put down for reference certain equations which have been taken (with proper adaptations) from Mr. W. M. Hicks' paper on Toroidal Functions,* and which will be useful hereafter. The equations are

$$\left.\begin{aligned} &(n+2)P_{n+2} - (2n+3)\nu P_{n+1} + (n+1)P_n = 0 \\ &\frac{\nu^2-1}{n+1}P_n' = P_{n+1} - \nu P_n \\ &\frac{\nu^2-1}{n}P_n' = \nu P_n - P_{n-1} \end{aligned}\right\} \ldots.(A),$$

which are satisfied also by the Q functions, and

$$\left.\begin{aligned} &P_{n+1}Q_n - P_n Q_{n+1} = \frac{1}{n+1} \\ &P_n' Q_n - P_n Q_n' = \frac{1}{\nu^2-1} \\ &P_n' Q_{n+1}' - P_{n+1}' Q_n' = \frac{n+1}{\nu^2-1} \end{aligned}\right\} \ldots\ldots\ldots(B),$$

* *Phil. Trans.* 1881, Arts. 7 and 9.

the accents denoting differentiation with respect to ν. They may be at once deduced (as in Mr. Hicks' paper) from the values of P_n and Q_n in terms of definite integrals, viz.

$$P_n = \frac{1}{\pi}\int_0^\pi \{\nu + \sqrt{(\nu^2-1)}\,\cos\theta\}^n\, d\theta,$$

$$Q_n = \int_0^\infty \frac{d\theta}{\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}^{n+1}},$$

whence $\quad Q_0 = \frac{1}{2}\log\frac{\nu+1}{\nu-1}, \quad Q_1 = \nu Q_0 - 1.$

Also, since $\nu > 1$,

$$Q_n = \frac{1}{\nu^{n+1}}\sum_0^\infty \frac{(2r+1)(2r+2)\ldots(n+2r)}{(2r+1)(2r+3)\ldots(2n+2r+1)}\,\frac{1}{\nu^{2r}}.$$

5. Before entering into the discussion of any physical problems it will be convenient to obtain the expansions of r, rx, and $r\rho$ in terms of harmonics. Now, if ϕ be any potential function such that $r\phi$ is finite at the origin, this quantity can, for points not lying along the lines $\nu = 1$, be expanded in a series of the form $\Sigma A_n t_n^{\,m}(\nu)\, T_n^{\,m}(\mu)\Sigma\sin(m\theta+\alpha_m)$; hence r and rx can be expanded in a series of zonal harmonics, and $r\rho\sin\theta$ in a series of tesseral harmonics.

(i) *Expansion of* r.

$$\frac{r}{c} = (\nu^2+\mu^2-1)^{-\frac{1}{2}} = \Sigma A_{2n} Q_{2n}(\nu)\, P_{2n}(\mu);$$

therefore

$$\frac{2A_{2n}Q_{2n}}{4n+1} = \frac{1}{\nu}\int_{-1}^1 \left\{1 + \ldots H_s\frac{(1-\mu^2)^s}{\nu^{2s}} + \ldots\right\} P_{2n}\, d\mu,$$

where (as throughout this paper) H_s is used to denote the coefficient of x^s in the expansion of $(1-x)^{-\frac{1}{2}}$.

Let $\qquad U_s = \int_{-1}^1 (1-\mu^2)^s\, P_{2n}\, d\mu;$

therefore

$$2n(2n+1)\,U_s = -\int_{-1}^1 (1-\mu^2)^s \frac{d}{d\mu}(1-\mu^2)\frac{dP_{2n}}{d\mu}\, d\mu$$

$$= -2s\int_{-1}^1 \mu(1-\mu^2)^s \frac{dP_{2n}}{d\mu}\, d\mu$$

$$= 2s\int_{-1}^1 \{(1-\mu^2)^s - 2s\mu^2(1-\mu^2)^{s-1}\}\, P_{2n}\, d\mu;$$

therefore $$U_s = \frac{2s^2}{(s-n)(2s+2n+1)} U_{s-1}$$

$$= \frac{2^r s^2 (s-1)^2 \ldots (s-r+1)^2 U_{s-r}}{(s-n)(s-n-1)\ldots(s-n-r+1)(2s+2n+1)(2s+2n-1)\ldots(2s+2n-2r+3)}.$$

Now

$$U_n = (-1)^n \int_{-1}^{1} \mu^{2n} P_{2n} d\mu = \frac{(-1)^n 2^{n+1} n!}{(2n+1)(2n+3)\ldots(4n+1)};$$

therefore, if $$s = n + r,$$

$$H_{n+r} U_{n+r} = \frac{2(-1)^n 1.3\ldots(2n+2r-1)(n+1)(n+2)\ldots(n+r)}{r!(2n+1)(2n+3)\ldots(4n+2r+1)}$$

$$= \frac{2H_n (-1)^n (2r+1)(2r+2)\ldots(2n+2r)}{(2r+1)(2r+3)\ldots(4n+2r+1)};$$

therefore $$\frac{2A_{2n} Q_{2n}}{4n+1} = 2(-1)^n H_n Q_{2n};$$

therefore $$\frac{r}{c} = \Sigma (-1)^n (4n+1) H_n Q_{2n} P_{2n} \ldots\ldots\ldots\ldots (6).$$

(ii) *Expansion of rx.*

$$\frac{rx}{c^2} = \frac{\mu\nu}{(\mu^2+\nu^2-1)^{\frac{3}{2}}} = \Sigma A_{2n+1} Q_{2n+1}(\nu) P_{2n+1}(\mu);$$

therefore

$$\frac{2A_{2n+1} Q_{2n+1}}{4n+3} = \frac{1}{\nu^2} \int_{-1}^{1} \mu \left\{ 1 + \ldots (2s+1) H_s \frac{(1-\mu^2)^s}{\nu^{2s}} \right\} P_{2n+1} d\mu.$$

Let $$U_s = \int_{-1}^{1} \mu (1-\mu^2)^s P_{2n+1} d\mu,$$

therefore

$$(2n+1)(2n+2) U_s = -\int_{-1}^{1} \mu (1-\mu^2)^s \frac{d}{d\mu} (1-\mu^2) \frac{dP_{2n+1}}{d\mu} d\mu$$

$$= (2s+1) \int_{-1}^{1} (1-\mu^2)^{s+1} \frac{dP_{2n+1}}{d\mu} - 2s \int_{-1}^{1} (1-\mu^2)^s \frac{dP_{2n+1}}{d\mu} d\mu$$

$$= (2s+1)(2s+2) U_s - 2^2 s^2 U_{s-1};$$

therefore $$U_s = \frac{2s^2}{(s-n)(2s+2n+3)} U_{s-1}$$

$$= \frac{2^r s^2 (s-1)^2 \ldots (s-r+1)^2 U_{s-r}}{(s-n)(s-n-1)\ldots(s-n-r+1)(2s+2n+3)(2s+2n+1)\ldots(2s+2n-2r+5)}.$$

Now $$U_n = (-)^n \int_{-1}^{1} \mu^{2n+1} P_{2n+1} d\mu$$

$$= \frac{(-1)^n 2^{n+1} n!}{(2n+3)(2n+5)\ldots(4n+3)};$$

therefore, if $$s = n + r,$$

$$(2n+2r+1) H_{n+r} U_{n+r}$$

$$= \frac{2(-1)^n 1.3\ldots(2n+2r+1)(n+1)(n+2)\ldots(n+r)}{r!(2n+3)(2n+5)\ldots(4n+2r+3)}$$

$$= \frac{2(-1)^n (2n+1) H_n (2r+1)(2r+2)\ldots(2n+2r+1)}{(2r+1)(2r+3)\ldots(4n+2r+3)};$$

therefore $$\frac{2A_{2n+1}Q_{2n+1}}{4n+3} = 2(-1)^n (2n+1) H_n Q_{2n+1};$$

therefore

$$\frac{rx}{c^2} = \Sigma (-1)^n (2n+1)(4n+3) H_n Q_{2n+1} P_{2n+1} \ldots\ldots (7).$$

(iii) *Expansion of* $r\rho$.

$$\frac{r\rho}{c^2} = \frac{\sqrt{(\nu^2-1)}\sqrt{(1-\mu^2)}}{(\nu^2+\mu^2-1)^{\frac{3}{2}}}.$$

We have shown that

$$\frac{\mu\nu}{(\nu^2+\mu^2-1)^{\frac{3}{2}}} = \Sigma A_{2n+1} Q_{2n+1} P_{2n+1},$$

where $$A_{2n+1} = (-)^n (2n+1)(4n+3) H_n;$$

therefore $$\frac{\mu}{(\mu^2+\nu^2-1)^{\frac{1}{2}}} = \Sigma A_{2n+1} P_{2n+1} \int_{\nu}^{\infty} Q_{2n+1} d\nu;$$

therefore $$\frac{\nu^2-1}{(\mu^2+\nu^2-1)^{\frac{3}{2}}} = \Sigma A_{2n+1} \frac{dP_{2n+1}}{d\mu} \int_{\nu}^{\infty} Q_{2n+1} d\nu.$$

But $$T_n^1 = (1-\mu^2)^{\frac{1}{2}} \frac{dP_n}{d\mu};$$

therefore, restoring the value of A_{2n+1},

$$\frac{r\rho}{c^2} = \frac{1}{\sqrt{(\nu^2-1)}} \Sigma (-1)^n (2n+1)(4n+3) H_n T^1_{2n+1} \int_\nu^\infty Q_{2n+1} d\nu \ldots (8).$$

We may put this result into another form, for

$$\int_\nu^\infty Q_n d\nu = -\frac{(\nu^2-1)\, Q'_n}{n(n+1)} = -\frac{\sqrt{(\nu^2-1)}\, t^1_n}{n(n+1)};$$

therefore

$$\frac{r\rho}{c^2} = -\Sigma (-1)^n \frac{4n+3}{2n+2} H_n t^1_{2n+1} T^1_{2n+1} \ldots\ldots (9).$$

(iv) *To find the value of* $\int_{-1}^{1} \frac{P_n d\mu}{(1-\mu^2)^{\frac{1}{2}}}$.

$$\Sigma x^n \int_{-1}^{+1} \frac{P_n d\mu}{(1-\mu^2)^{\frac{1}{2}}} = \int_0^\pi \frac{d\theta}{(1-2x\cos\theta+x^2)^{\frac{1}{2}}}$$

$$= 2\int_0^{\frac{1}{2}\pi} \frac{d\chi}{(1+x)\left\{1-\frac{4x\cos^2\chi}{(1+x)^2}\right\}^{\frac{1}{2}}}$$

$$= 2\int_0^{\frac{1}{2}\pi} \frac{d\psi}{(1-x^2\cos^2\psi)^{\frac{1}{2}}}$$

$$= \pi \Sigma H^2_m x^{2m};$$

therefore, when n is odd the integral is zero, and when n is even and equal to $2m$ its value is πH^2_m.

(v) *To find the value of* $\int_{-1}^{1} \frac{\mu P_n d\mu}{(1-\mu^2)^{\frac{1}{2}}}$.

$$\Sigma x^n \int_{-1}^{1} \frac{\mu P_n d\mu}{(1-\mu^2)^{\frac{1}{2}}} = -\int_0^\pi \frac{\cos\theta\, d\theta}{(1+2x\cos\theta+x^2)^{\frac{1}{2}}}$$

$$= x\int_0^\pi \frac{\sin^2\theta}{(1+2x\cos\theta+x^2)^{\frac{3}{2}}}$$

$$= 2x\int_0^{\frac{1}{2}\pi} \frac{\sin^2 2\phi\, d\phi}{(1+x)^3\left\{1-\frac{4x}{(1+x)^2}\sin^2\phi\right\}^{\frac{3}{2}}};$$

putting

$$\tan\theta = \frac{\sin 2\phi}{x+\cos 2\phi},$$

the integral is

$$= x\int_0^\pi \frac{\sin^2\theta\, d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}}};$$

hence, when n is even the integral is zero, and when n is odd and equal to $2m+1$ its value is

$$\pi H_m H_{m+1} = \frac{\pi H_m^2 (2m+1)}{2m+2}.$$

6. Having obtained these preliminary results, we shall proceed to find the potential of a charged conductor in the following cases:

(α) *When the conductor is uninfluenced by electrical forces.*

Let $$V = \frac{c}{r} \Sigma A_n \frac{P_n(\nu)}{P_n(\gamma)} P_n(\mu),$$

where $\gamma = \cosh\beta$ is the value of ν at the surface; hence if V_0 be the constant value of V at the surface,

$$\Sigma A_n P_n(\mu) = V_0 c \Sigma (-1)^n (4n+1) H_n Q_{2n}(\gamma) P_{2n}(\mu) \quad \text{by (6)};$$

therefore

$$V = \frac{V_0 c}{r} \Sigma (-1)^n (4n+1) H_n \frac{Q_{2n}(\gamma)}{P_{2n}(\gamma)} P_{2n}(\nu) P_{2n}(\mu).$$

(β) *When the conductor is placed in a uniform field of force parallel to the axis.*

If A be the force,

$$V - V_0 = -Ax + \frac{c^2}{r} \Sigma A_n \frac{P_n(\nu)}{P_n(\gamma)} P_n(\mu);$$

therefore at the surface

$$\Sigma A_n P_n(\mu) = A \Sigma (-1)^n (2n+1)(4n+3) H_n Q_{2n+1}(\gamma) P_{2n+1}(\mu)$$

by (7); therefore

$$V - V_0 = -Ax + \frac{Ac^2}{r} \times \Sigma (-1)^n (2n+1)(4n+3) H_n \frac{Q_{2n+1}(\gamma)}{P_{2n+1}(\gamma)} P_{2n+1}(\nu) P_{2n+1}(\mu).$$

(γ) *When the conductor is placed in a uniform field of force perpendicular to a plane containing the axis.*

We may take the plane of xy as the plane in question; hence, if B be the force,

$$V - V_0 = -B\rho \sin\theta + \frac{c^2 \sin\theta}{r} \Sigma A_n \frac{T_n^1(\nu)}{T_n^1(\gamma)} T_n^1(\mu);$$

therefore at the surface

$$\Sigma A_n T_n^1(\mu) = -\Sigma(-1)^n \frac{4n+3}{2n+2} H_n t^1_{2n+1}(\gamma)\, T^1_{2n+1}(\mu)$$

by (9); therefore

$$V - V_0 = -Bz + \frac{c^2 \sin\theta}{r} \Sigma \frac{(4n+3) H_n\, t^1_{2n+1}(\gamma)}{(2n+2) H_n T^1_{2n+1}(\gamma)} T^1_{2n+1}(\nu) T^1_{2n+1}(\mu).$$

(δ) *When the surface is under the action of a charge* e *at the focus.*

$$V - V_0 = \frac{e}{FP} + \frac{1}{r} \Sigma A_n \frac{P_n(\nu)}{P_n(\gamma)} P_n(\mu);$$

therefore at the surface

$$\Sigma A_n P_n \mu = -\frac{e}{\gamma - \mu} = -e\Sigma(2n+1)\, Q_n(\gamma)\, P_n(\mu);$$

therefore

$$V - V_0 = \frac{e}{FP} - \frac{e}{r} \Sigma(2n+1) \frac{Q_n(\gamma)}{P_n(\gamma)} P_n(\nu) P_n(\mu).$$

If we invert these four results with respect to the centre, we shall obtain the potential within an ovary ellipsoid when the surface encloses:

(ε) *a charge at the centre;*

(ζ) *a system whose potential is proportional to* $\frac{x}{r^3}$;

(η) *a system whose potential is proportional to* $\frac{z}{r^3}$;

(θ) *a charge at the focus.*

The last result is given in Ferrers' *Spherical Harmonics*, p. 142.

7. *To find the attraction of the solid (supposed to be homogeneous) parallel to the axis.*

If σ be the density, we know that the x-component of the attraction is the potential of a surface distribution of matter of density σl, where l is the x-direction cosine of the normal to the surface; hence, if X, X' be the values of this component at an external and an internal point respectively, the surface conditions are

$$X - X' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

$$\frac{dX}{dn} - \frac{dX'}{dn} = 4\pi\sigma l \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Now $$l = \frac{px}{r^4}(2r^2 - a^2),$$

$$J = \frac{2pc^2}{r^4 \sinh 2\beta};$$

therefore (11) becomes

$$\frac{dX}{d\nu} - \frac{dX'}{d\nu} = -\frac{4\pi\sigma\gamma}{c^2}(2r^2 - a^2)x \quad \ldots\ldots\ldots(12).$$

Now (10) requires that

$$X = \frac{1}{r}\Sigma A_n \frac{P_n(\nu)}{P_n(\mu)} P_n(\mu),$$

and $$X' = \frac{1}{r}\Sigma A_n \frac{Q_n(\nu)}{Q_n(\gamma)} P_n(\mu);$$

therefore the left-hand side of (11)

$$= \frac{1}{r}\Sigma A_n \left\{\frac{P_n'(\gamma)}{P_n(\gamma)} - \frac{Q'_n(\gamma)}{Q_n(\gamma)}\right\} P_n(\mu)$$

$$= \frac{1}{r \sinh^2\beta}\Sigma \frac{A_n P_n(\mu)}{P_n(\gamma)\, Q_n(\gamma)}$$

by equations (B) of Art. 4.

Now $$r^3 x = \frac{c^4 \mu\gamma}{(\mu^2 + \gamma^2 - 1)^{\frac{5}{2}}};$$

therefore, if $$u = \frac{\mu\gamma}{(\mu^2 + \gamma^2 - 1)^{\frac{3}{2}}};$$

$$\frac{r^3 x}{c^4} = \frac{1}{3\gamma^2}\left(u - \gamma\frac{du}{d\gamma}\right);$$

therefore

$$(2r^2 - a^2) rx = -\frac{c^4}{3\gamma^2}\left(u + 2\gamma\frac{du}{d\gamma}\right)$$

$$= -\frac{c^4}{3\gamma^2}\Sigma(-1)^n(2n+1)(4n+3) \times H_n\{2\gamma Q'_{2n+1}(\gamma) + Q_{2n+1}(\gamma)\} P_{2n+1}(\mu)$$

by (7); therefore $A_{2n} = 0$,

and

$$A_{2n+1} = \frac{4\pi\sigma c^2 \sinh^2\beta}{3\cosh\beta}(-1)^n(2n+1)(4n+3) \times H_n P_{2n+1}(\gamma)\, Q_{2n+1}(\gamma)\{2\gamma Q'_{2n+1}(\gamma) + Q_{2n+1}(\gamma)\}.$$

8. *To find the attraction perpendicular to the axis.*

Let R, R' be the attractions at an external and internal point respectively; Z, Z' the z-components; then at the surface

$$\frac{dZ}{d\nu} - \frac{dZ'}{d\nu} = -\frac{4\pi\sigma\gamma}{c^2}(2r^2 - b^2)z;$$

therefore

$$\frac{dR}{d\nu} - \frac{dR'}{d\nu} = -\frac{4\pi\sigma\gamma}{c^2}(2r^2 - b^2)\rho \quad \ldots\ldots\ldots\ldots (13).$$

Now

$$r^3\rho = \frac{c^4\sqrt{(\gamma^2-1)}\sqrt{(1-\mu^2)}}{(\mu^2+\gamma^2-1)^{\frac{5}{2}}};$$

therefore, if

$$u = \frac{\sqrt{(1-\mu^2)}\sqrt{(\gamma^2-1)}}{(\mu^2+\gamma^2-1)^{\frac{3}{2}}},$$

$$\frac{r^3\rho}{c^4} = \tfrac{1}{3}\left(\frac{u}{\gamma^2-1} - \frac{1}{\gamma}\frac{du}{d\gamma}\right),$$

and

$$\frac{b^2 r\rho}{c^4} = \frac{u}{\gamma^2-1};$$

therefore

$$(2r^2 - b^2)\rho r = -\tfrac{1}{3}c^4\left(\frac{u}{\gamma^2-1} + \frac{2}{\gamma}\frac{du}{d\gamma}\right)$$

$$= \frac{c^4}{3\sqrt{(\gamma^2-1)}}\Sigma(-1)^n(2n+1)(4n+3)H_n T^1_{2n+1}(\mu) \times \left\{\frac{2}{\gamma}Q_{2n+1}(\gamma) - \frac{1}{\gamma^2-1}\int_\gamma^\infty Q_{2n+1}\,d\nu\right\}.$$

But

$$\int_\gamma^\infty Q_{2n+1}d\nu = -\frac{(\gamma^2-1)Q'_{2n+1}}{(2n+1)(2n+2)};$$

therefore the right-hand side of (13)

$$= -\frac{4\pi\sigma c^2}{3\sinh\beta}\Sigma(-1)^n(2n+1)(4n+3)H_n \times \left\{2Q_{2n+1}(\gamma) + \frac{\gamma Q'_{2n+1}(\gamma)}{(2n+1)(2n+2)}\right\} T^1_{2n+1}(\mu).$$

We must therefore (omitting the affix 1) assume

$$R = \frac{1}{r} \Sigma A_{2n+1} \frac{T_{2n+1}(\nu)}{T_{2n+1}(\gamma)} T_{2n+1}(\mu),$$

$$R' = \frac{1}{r} \Sigma A_{2n+1} \frac{t_{2n+1}(\nu)}{t_{2n+1}(\gamma)} T_{2n+1}(\mu);$$

therefore

$$\frac{dR}{d\nu} - \frac{dR'}{d\nu} = \frac{1}{r} \Sigma A_{2n+1} \left\{ \frac{T_{2n+1}(\nu)}{T_{2n+1}(\gamma)} - \frac{t'_{2n+1}(\nu)}{t_{2n+1}(\gamma)} \right\} T_{2n+1}(\mu),$$

the accents denoting differentiation.

Now $T_n = \sqrt{(\gamma^2 - 1)}\, P_n'$;

therefore $T_n' = \dfrac{1}{\sqrt{(\gamma^2 - 1)}} \{3\gamma P_n' + n(n+1) P_n\}$;

therefore $t_n T_n' - t'_n T_n = n(n+1)(P_n Q'_n - P_n' Q_n)$

$$= -\frac{n(n+1)}{\gamma^2 - 1};$$

therefore

$$A_{2n+1} = \frac{4\pi\sigma c^2 \sinh\beta}{3} (-1)^n \frac{4n+3}{2n+2} H_n T_{2n+1}(\gamma)\, t_{2n+1}(\gamma) \left\{ 2 Q_{2n+1}(\gamma) + \frac{\gamma Q'_{2n+1}(\gamma)}{(2n+1)(2n+2)} \right\}.$$

9. When the potential is symmetrical about the axis of revolution, there exists a current function ψ, such that the lines of intersection of the surface $\psi = C$, with the plane $z = y \tan\theta$ determine the lines of force or lines of flow, according as ϕ is an electric or a velocity potential. This function satisfies the equation

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{d\rho^2} - \frac{1}{\rho} \frac{d\psi}{d\rho} = 0 \quad \ldots\ldots\ldots\ldots (14),$$

and is connected with ϕ by the equations

$$\left.\begin{aligned} u = \frac{d\phi}{dx} &= \frac{1}{\rho} \frac{d\psi}{d\rho} \\ v = \frac{d\phi}{d\rho} &= -\frac{1}{\rho} \frac{d\psi}{dx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (15).$$

We shall now proceed to transform these equations.

Let $$\psi = U\sqrt{\rho},$$

then (14) becomes

$$\frac{d^2U}{dx^2} + \frac{d^2U}{d\rho^2} = \frac{3U}{4\rho^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Put $$x + \iota\rho = f(\xi + \iota\eta),$$

then (16) becomes

$$\frac{d^2U}{d\xi^2} + \frac{d^2U}{d\eta^2} = \frac{3U}{4J^2\rho^2} \quad \ldots\ldots\ldots\ldots\ldots (17).$$

Putting, as before, $U = W\sqrt{(uv)}$, where u is a function of ξ only, and v a function of η, (17) becomes

$$\frac{1}{u}\frac{d}{d\xi}\left(u\frac{dW}{d\xi}\right) + \frac{1}{v}\frac{d}{d\eta}\left(v\frac{dW}{d\eta}\right) + \left(\frac{u''}{2u} - \frac{u'^2}{4u^2} + \frac{v''}{2v} - \frac{v'^2}{4v^2}\right)W = \frac{3W}{4J^2\rho^2} \ldots\ldots (18).$$

In the next place let p and q be the velocities along the normals to the surfaces η and ξ, drawn in the direction in which these quantities increase. Then

$$p = J\frac{d\phi}{d\eta} = -lu - mv,$$

$$q = J\frac{d\phi}{d\xi} = \quad lv - mu,$$

also

$$Jl = \frac{d\eta}{dx} = -\frac{d\xi}{d\rho},$$

$$Jm = \frac{d\eta}{d\rho} = \quad \frac{d\xi}{dx};$$

therefore

$$p\rho = -\left(\frac{d\psi}{d\xi}\cdot\frac{d\xi}{d\rho} + \frac{d\psi}{d\eta}\cdot\frac{d\eta}{d\rho}\right)l + \left(\frac{d\psi}{d\xi}\cdot\frac{d\xi}{dx} + \frac{d\psi}{d\eta}\cdot\frac{d\eta}{dx}\right)m;$$

therefore

$$p = \frac{J}{\rho}\frac{d\psi}{d\xi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

$$q\rho = -\left(\frac{d\psi}{d\xi}\cdot\frac{d\xi}{dx} + \frac{d\phi}{d\eta}\cdot\frac{d\eta}{dx}\right)l - \left(\frac{d\psi}{d\xi}\cdot\frac{d\xi}{d\rho} + \frac{d\psi}{d\eta}\cdot\frac{d\eta}{d\rho}\right)m,$$

$$q = -\frac{J}{\rho}\frac{d\psi}{d\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

10. To adapt (18) to the present case, we must make the same substitutions as in Art. 3, and we shall find that W can be expressed in a series of the form $\Sigma X_n Y_n$, when X_n and Y_n respectively satisfy the equations

$$\frac{d}{d\mu}(1-\mu^2)\frac{dX_n}{d\mu} + \left\{n(n+1) - \frac{1}{1-\mu^2}\right\} X_n = 0,$$

and

$$\frac{d}{d\nu}(1-\nu^2)\frac{dY_n}{d\nu} + \left\{n(n+1) - \frac{1}{1-\nu^2}\right\} Y_n = 0,$$

where n is zero or an integer; hence, when n is not zero,

$$X_n = \quad T_n^1(\mu),$$

$$Y_n = A_n T_n^1(\nu) + B_n t^1{}_n(\nu),$$

and

$$\psi = \frac{\rho}{r}\Sigma_1^\infty \{A_n T_n^1(\nu) + B_n t^1{}_n(\nu)\} T_n^1(\mu),$$

but when $n = 0$,

$$X_0 = \frac{A\mu + B}{\sqrt{(1-\mu^2)}},$$

$$Y_0 = \frac{A'\nu + B}{\sqrt{(\nu^2-1)}};$$

therefore

$$\frac{\rho}{r} X_0 Y_0 = (A\mu + B)(A'\nu + B')\frac{r}{c} = Z \text{ (say)}.$$

Referring to the values of p and q, we see that

$$p = \quad \frac{Jc}{r^2\sqrt{(\nu^2-1)}}\frac{d\psi}{d\mu},$$

$$q = -\frac{Jc}{r^2\sqrt{(1-\mu^2)}}\frac{d\psi}{d\nu}.$$

Hence, the velocities due to the term Z will be infinite at the foci unless $\frac{dZ}{d\mu}$ and $\frac{dZ}{d\nu}$ either vanish or are infinitesimal quantities of the third or higher orders in the neighbourhood of these points; these conditions cannot be satisfied unless A, B, A' and B' are zero. It follows therefore that the terms X_0, Y_0 do not appear in the expression for ψ when we are dealing with space outside the surfaces. On the other hand when μ is equal to $+1$ or -1, $\frac{dZ}{d\nu}$ will vanish if $B' = 0$;

the conclusion therefore is, that for space outside the surface the proper form of ψ is

$$\frac{\rho}{r}\,\Sigma_1^\infty A_n T_n^1(\nu)\,T_n^1(\mu),$$

and for space within the surface

$$\psi = \frac{r}{c}(A\mu + B)\,\nu + \frac{\rho}{r}\,\Sigma_1^\infty A_n t_n^1(\nu)\,T_n^1(\mu).$$

11. *To find ψ when the solid is moving with velocity U parallel to the axis in an infinite liquid.*

Let $$\psi = \frac{\rho}{r}\,\Sigma_1^\infty A_n \frac{T_n^1(\nu)}{T_n^1(\gamma)}\,T_n^1(\mu),$$

at the surface $$\psi = \tfrac{1}{2}U\rho^2;$$

therefore $$\Sigma A_n T_n^1 \mu = \tfrac{1}{2}Ur\rho$$

$$= -\frac{Uc^2}{2}\,\Sigma(-1)^n \frac{4n+3}{2n+2}\,t^1_{2n+1}(\gamma)\,T^1_{2n+1}(\mu);$$

therefore

$$\psi = -\frac{Uc^2\rho}{2r}\,\Sigma_0^\infty \frac{(-1)^n(4n+3)\,t^1_{2n+1}(\gamma)}{(2n+2)\,T^1_{2n+1}(\gamma)}\,T^1_{2n+1}(\nu)\,T^1_{2n+1}(\mu).$$

Part II.

12. We shall now consider the surfaces which are the inverses of planetary ellipsoids.

Taking the axis of z as the axis of revolution, Laplace's equation is

$$\frac{d^2\phi}{dz^2} + \frac{d^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{d\phi}{d\rho} + \frac{1}{\rho^2}\frac{d^2\phi}{d\theta^2} = 0 \ldots\ldots\ldots (21).$$

Putting $$\rho + \iota z = f(\xi + \iota\eta),$$

and employing the same method of transformation as before, we shall be conducted to equation (5).

In the present case

$$\rho + \iota z = c\sec(\xi + \iota\eta),$$

and $$\frac{1}{J^2\rho^2} = \sec^2\xi - \operatorname{sech}^2\eta.$$

If, therefore, we put

$$u = \cos\xi, \quad v = \cosh\eta,$$
$$\mu = \sin\xi, \quad \nu = \sinh\eta,$$

we shall finally obtain

$$\phi = \frac{1}{r^{\frac{1}{2}}} \Sigma X_n Y_n \Sigma \sin(m\theta + \alpha_m),$$

where X_n, Y_n respectively satisfy the equations

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dX_n}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\} X_n = 0,$$

$$\frac{d}{d\nu}\left\{(1+\nu^2)\frac{dY_n}{d\nu}\right\} - \left\{n(n+1) - \frac{m^2}{1+\nu^2}\right\} Y_n = 0.$$

13. When ϕ is symmetrical about the axis of z, so that m is zero, X_n is of the form $A_n P_n(\mu)$, and Y_n of the form $A_n P_n(\iota\nu) + B_n Q_n(\iota\nu)$; now we know that

$$Q_n(\iota\nu) = \frac{1}{(-1)^{\frac{1}{2}(n+1)}} \int_0^\infty \frac{d\theta}{\{\nu + \sqrt{(\nu^2+1)}\cosh\theta\}^{n+1}},$$

and

$$P_n(\iota\nu) = \frac{(-1)^{\frac{1}{2}n}}{\pi} \int_0^\pi \{\nu + \sqrt{(\nu^2+1)}\cos\theta\}^n \, d\theta.$$

Hence if we put $q_n(\nu)$, and $p_n(\nu)$ for the two definite integrals respectively,

$$Q_n(\iota\nu) = (-1)^{-\frac{1}{2}(n+1)} q_n(\nu),$$
$$P_n(\iota\nu) = (-1)^{\frac{1}{2}n} p_n(\nu),$$

so that

$$\phi = \frac{1}{r} \Sigma \{A_n q_n(\nu) + B_n p_n(\nu)\} P_n(\mu).$$

Also

$$q_0 = \cot^{-1}\nu,$$
$$q_1 = 1 - \nu\cot^{-1}\nu,$$
$$p_0 = 1,$$
$$p_1 = \nu;$$

and equations (A) and (B) of Art. 4 become

$$\left.\begin{aligned} (n+2)\,q_{n+2} + (2n+3)\,\nu q_{n+1} - (n+1)\,q_n &= 0 \\ \frac{\nu^2+1}{n+1} \cdot \frac{dq_n}{d\nu} &= -q_{n+1} - \nu q_n \\ \frac{\nu^2+1}{n} \cdot \frac{dq_n}{d\nu} &= \nu q_n - q_{n-1} \end{aligned}\right\} \ldots\ldots (C),$$

which are also satisfied by $(-1)^n p_n$; and

$$\left.\begin{aligned} p_{n+1}q_n + q_{n+1}p_n &= \frac{1}{n+1} \\ p'_n q_n - q'_n p_n &= \frac{1}{\nu^2+1} \\ p'_n q'_{n+1} + q'_n p'_{n+1} &= \frac{(n+1)}{\nu^2+1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (D).$$

If we put $\cosh\theta = \sec\phi$,

$$q_n = \int_0^{\frac{1}{2}\pi} \frac{\cos^n\phi\, d\phi}{\{\nu\cos\phi + \sqrt{(\nu^2+1)}\}^{n+1}};$$

therefore

$$q_{2n}(0) = \tfrac{1}{2}\pi H_n,$$

$$q_{2n+1}(0) = \frac{1}{(2n+1)H_n}.$$

14. We must in the next place show that the q functions are suitable for space within the surface, and the p functions for space without it.

When $\nu = 0$ the surfaces double back and form a doubled plane with a circular orifice in the centre of radius c, and extending in all directions from the boundary of this circle to infinity. The conditions of continuity require* that $\frac{d\phi}{d\eta}$ and $\frac{d\phi}{d\xi}$ should be small quantities of the first order in the neighbourhood of this circle, and also that these quantities should either vanish or change sign in passing from one side of this plane to the other.

Let $$\phi = \frac{1}{r} q_n P_n;$$

therefore $$\frac{d\phi}{d\eta} = \cosh\eta \left(\frac{r\nu}{c^2} q_n + \frac{q'_n}{r}\right) P_n,$$

and $$\frac{d\phi}{d\xi} = \cos\xi \left(\frac{P'_n}{r} - \frac{r\mu}{c^2} P_n\right) q_n.$$

When n is odd $\frac{d\phi}{d\eta}$ is an infinitesimal of the first order at the circle and changes sign with μ, but it fulfils neither of these

* See page 195.

conditions when n is even. When n is even $\frac{d\phi}{d\xi}$ vanishes at the circle and changes sign with μ, but it fulfils neither of these conditions when n is odd. Therefore the q functions are not suitable for space outside the surface.

Next let $$\phi = \frac{1}{r} p_n P_n;$$

therefore $$\frac{d\phi}{d\eta} = \cosh\eta \left(\frac{p_n'}{r} + \frac{r\nu}{c^2} p_n\right) P_n,$$

and $$\frac{d\phi}{d\xi} = \cos\xi \left(\frac{P_n'}{r} - \frac{r\mu}{c^2} P_n\right) p_n \nu.$$

When n is even $\frac{d\phi}{d\eta}$ vanishes when $\nu = 0$, and when n is odd it changes sign with μ and vanishes when $\mu = 0$. When n is odd $\frac{d\phi}{d\xi}$ vanishes when $\nu = 0$, and when n is even it changes sign with μ and vanishes when $\mu = 0$. Hence the p functions are suitable for space without the surface; but they are not suitable for space within the surface, because they become infinite at the origin.

15. When the potential is unsymmetrical about the axis, the tesseral functions take the place of the p and q functions.

Now $$T_n^m(\iota\nu) = (1+\nu^2)^{\frac{1}{2}m} \frac{d^m P_n(\iota\nu)}{d\nu^m}$$

$$= (-1)^{\frac{1}{2}(n+m)} (1+\nu^2)^{\frac{1}{2}m} \frac{d^m p_n}{d\nu^m}.$$

Similarly, $t_n^m(\iota\nu) = (-1)^{-\frac{1}{2}(n-m+1)} (1+\nu^2) \frac{d^m q_n}{d\nu^m}$.

Hence if we respectively put $\mathfrak{T}_n^m(\nu)$ and $\mathfrak{t}_n^m(\nu)$ for the two factors involving ν,

$$T_n^m(\iota\nu) = (-1)^{\frac{1}{2}(n+m)} \mathfrak{T}_n^m(\nu),$$

and $$t_n^m(\iota\nu) = (-1)^{-\frac{1}{2}(n-m+1)} \mathfrak{t}_n^m(\nu);$$

and the potentials for space without and within the surface are respectively equal to

$$\frac{1}{r} \Sigma A_n \mathfrak{T}_n^m(\nu)\ T_n^m(\mu) \Sigma \sin(m\theta + \alpha_m),$$

and $$\frac{1}{r} \Sigma B_n \mathfrak{t}_n^m(\nu)\ \ T_n^m(\mu) \Sigma \sin(m\theta + \alpha_m).$$

16. We shall in the next place find the expansions of r, rz, and $r\rho$.

(i) *Expansion of r.*

$$\frac{r}{c} = (1 + \nu^2 - \mu^2)^{-\frac{1}{2}} = \Sigma A_n q_{2n} P_{2n};$$

therefore, when $\nu = 0$,

$$(1 - \mu^2)^{-\frac{1}{2}} = \tfrac{1}{2}\pi \Sigma A_n H P_{2n};$$

therefore
$$\frac{\pi A_n H_n}{4n+1} = \int_{-1}^{1} \frac{P_{2n} d\mu}{(1-\mu^2)^{\frac{1}{2}}} = \pi H_n^2,$$

by Art. 5, s. iv;

therefore
$$A_n = (4n+1)\, H_n;$$

and
$$(1 + \nu^2 - \mu^2)^{-\frac{1}{2}} = \Sigma (4n+1) H_n q_{2n} P_{2n} \ldots\ldots\ldots (22),$$

putting $\mu = 0$,

$$(1 + \nu^2)^{-\frac{1}{2}} = \Sigma (4n+1)(-1)^n H^2 q_{2n} \ldots\ldots\ldots (23).$$

The last series is convergent provided ν is not zero; for

$$q_n = \int_0^{\frac{1}{2}\pi} \frac{\cos^n \theta \, d\theta}{\{\nu \cos\theta + \sqrt{(\nu^2+1)}\}^{n+1}};$$

therefore
$$q_{n+1} < \frac{q_n}{\nu + \sqrt{(\nu^2+1)}};$$

therefore
$$\frac{q_{n+1}}{q_n} < \varepsilon^{-\eta};$$

therefore

$$\frac{(4n+5) H^2_{n+1} q_{2n+2}}{(4n+1) H_n^2 q_{2n}} < \left\{1 + \frac{1}{(4n+1)(2n+2)^2}\right\} \varepsilon^{-2\eta};$$

therefore the series (23) is convergent except when $\eta = 0$, and the series (22) is convergent for all values of ν, except when $\nu = 0$ and $\mu = 1$.

(ii) *Expansion of rz.*

$$\frac{rz}{c^2} = \frac{\mu\nu}{(1+\nu^2-\mu^2)^{\frac{3}{2}}} = \Sigma A_{2n+1} P_{2n+1} q_{2n+1},$$

therefore
$$\frac{\mu}{(1-\mu^2)^{\frac{1}{2}}} = \Sigma A_{2n+1} P_{2n+1} \int_0^\infty q_{2n+1} d\nu.$$

But

$$q'_{n+1} + q'_{n-1} = \frac{1}{\nu^2+1}\{(n+1)\,\nu q_{n+1} - n\nu q_{n-1} - (2n+1)\,q_n\}$$

$$= -(2n+1)\,q_n,$$

by equations (C) of Art. 13

therefore $$(2n+1)\int_0^\infty q_n d\nu = q_{n+1}(0) + q_{n-1}(0);$$

therefore $$(4n+3)\int_0^\infty q_{2n+1} d\nu = \tfrac{1}{2}\pi\,(H_{n+1} + H_n);$$

therefore $$\int_0^\infty q_{2n+1} d\nu = \frac{\pi H_n}{4\,(n+1)};$$

therefore $$\frac{\pi H_n A_{2n+1}}{2\,(4n+3)\,(n+1)} = \int_{-1}^{1} \frac{\mu P_{2n+1}}{\sqrt{(1-\mu^2)}}\,d\mu$$

$$= \frac{\pi H_n^2\,(2n+1)}{2\,(n+1)},$$

by Art. 5 (s. v); therefore

$$A_{2n+1} = H_n\,(2n+1)\,(4n+3);$$

therefore

$$\frac{\mu\nu}{(1+\nu^2-\mu^2)^{\frac{3}{2}}} = \Sigma\,(2n+1)\,(4n+3)\,H_n q_{2n+1} P_{2n+1} \ldots\ldots (24).$$

(iii) *Expansion of* $r\rho$.

$$\frac{r\rho}{c^2} = \frac{\sqrt{(\nu^2+1)}\,\sqrt{(1-\mu^2)}}{(1+\nu^2-\mu^2)^{\frac{3}{2}}}.$$

Now, by (24)

$$\frac{\mu}{(1+\nu^2-\mu^2)^{\frac{1}{2}}} = \Sigma\,(4n+3)\,(2n+1)\,H_n P_{2n+1}\int_\nu^\infty q_{2n+1}\,d\nu \ldots. (25).$$

Differentiating with respect to μ,

$$\frac{\sqrt{(\nu^2+1)}\,\sqrt{(1-\mu^2)}}{(1+\nu^2-\mu^2)^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{(\nu^2+1)}}\Sigma\,(4n+3)\,(2n+1)\,H_n T^n{}_{2n+1}\int_\nu^\infty q_{2n+1} d\nu \ldots (26).$$

But $$\frac{1}{\sqrt{(\nu^2+1)}}\int_\nu^\infty q_{2n+1}d\nu = -\frac{\sqrt{(1+\nu^2)}\,q'_{2n+1}}{(2n+1)(2n+2)}$$

$$= -\frac{\mathfrak{t}^1_{2n+1}}{(2n+1)(2n+2)},$$

Therefore the left-hand side of (26)

$$= -\Sigma\frac{4n+3}{2n+2}H_n\mathfrak{t}^1_{2n+1}T^1_{2n+1} \quad \ldots\ldots\ldots\ldots(27).$$

17. By means of the results of the last article the potential in the cases (α), (β), (γ) of Art. 6 are easily seen to be

$$(\alpha') \qquad V = V_0\frac{c}{r}\Sigma(4n+1)H_n\frac{q_{2n}(\gamma)}{p_{2n}(\gamma)}p_{2n}(\nu)P_{2n}(\mu),$$

$$(\beta') \quad V - V = -Az + \frac{Ac^2}{r}\Sigma(4n+3)(2n+1)H_n\frac{q_{2n+1}(\gamma)}{p_{2n+1}(\gamma)} \times p_{2n+1}(\nu)P_{2n+1}(\mu),$$

$$(\gamma') \quad V - V_0 = -\beta\rho\sin\theta - \frac{Bc^2\sin\theta}{r}\Sigma\frac{(4n+3)\mathfrak{t}^1_{2n+1}(\gamma)}{(2n+2)\mathfrak{T}^1_{2n+1}(\gamma)} \times H_n\mathfrak{T}^1_{2n+1}(\nu)T^1_{2n+1}(\mu),$$

and the current function when the solid is moving parallel to z with velocity V is

$$\psi = -\frac{Vc^2\rho}{2r}\Sigma\frac{(4n+3)\mathfrak{t}^1_{2n+1}(\gamma)}{(2n+2)\mathfrak{T}^1_{2n+1}(\gamma)}H_n\mathfrak{T}^1_{2n+1}(\nu)T^1_{2n+1}(\mu),$$

where $\gamma = \sinh\beta$ is the value of ν at the surface, and the summations extend from ∞ to 0.

18. *To find the potential when the surface is under the action of a concentric circular wire of radius c, which is situated in the plane xy, and charged with electricity of line-density σ per unit of length.*

The potential of the wire is

$$V = 2\int_0^\pi \frac{c\sigma\,d\psi}{(c^2+r^2-2cr\cos\theta\cos\psi)^{\frac{1}{2}}}$$

$$= \frac{2c\sigma}{r}\int_0^\pi \frac{d\psi}{\{2+\nu^2-\mu^2-2\sqrt{(\nu^2+1)}\sqrt{(1-\mu^2)}\cos\psi\}^{\frac{1}{2}}} \quad \ldots(28).$$

At a point on the axis of z, $\mu = 1$; therefore

$$V = \frac{2c\sigma}{r} \cdot \frac{\pi}{(1+\nu^2)^{\frac{1}{2}}}$$

$$= \frac{2\pi c\sigma}{r} \Sigma (4n+1)(-1)^n H_n^2 q_{2n},$$

by (23) therefore

$$V = \frac{2\pi c\sigma}{r} \Sigma (4n+1)(-1)^n H_n^2 q_{2n} P_{2n} \quad \ldots\ldots\ldots (29).$$

The potential of the electric field is therefore

$$\phi - \phi_0 = \frac{1}{r} \Sigma A_n \frac{p_n(\nu)}{p_n(\gamma)} P_n(\mu) + \frac{2\pi c\sigma}{r} \Sigma (4n+1)(-1)^n H_n^2 q_{2n}(\nu) P_{2n}(\mu);$$

therefore $$A_{2n+1} = 0,$$

and $$A_{2n} = -2\pi c\sigma (4n+1)(-1)^n H_n^2 q_{2n}(\gamma);$$

therefore

$$\phi - \phi_0 = V - \frac{2\pi c\sigma}{r} \Sigma (4n+1)(-1)^n H_n^2 \frac{q_{2n}(\gamma)}{p_{2n}(\gamma)} p_{2n}(\nu) P_{2n}(\mu).$$

If we invert this result with respect to the centre, we shall obtain the potential within a planetary ellipsoid which surrounds a charged circular wire, passing through the foci of the generating ellipse.

19. Comparing (28) and (29) we see that

$$\pi \Sigma (4n+1)(-1)^n H_n^2 q_{2n}(\nu) P_{2n}(\mu)$$

$$= \int_0^\pi \frac{d\psi}{\{2 + \nu^2 - \mu^2 - 2\sqrt{(\nu^2+1)}\sqrt{(1-\mu^2)}\cos\psi\}^{\frac{1}{2}}}.$$

If we put $\nu = 0$, the left-hand side

$$= \tfrac{1}{2}\pi^2 \Sigma (4n+1)(-1)^n H_n^3 P_{2n}(\mu).$$

The right-hand side

$$= \int_0^\pi \frac{d\psi}{\{2 - \mu^2 - 2\sqrt{(1-\mu^2)}\cos\psi\}^{\frac{1}{2}}}$$

$$= \frac{2}{1+k'} \int_0^{\frac{1}{2}\pi} \frac{d\chi}{\left\{1 - \frac{4\chi}{(1+k')^2}\cos^2\chi\right\}^{\frac{1}{2}}}$$

$$= 2 \int_0^{\frac{1}{2}\pi} \frac{d\chi}{(1 - k'^2\cos^2\chi)^{\frac{1}{2}}},$$

where $$k' = \sqrt{(1-\mu^2)};$$

therefore $$K' = \tfrac{1}{4}\pi^2 \Sigma (4n+1)(-1)^n H_n {}^3P_{2n}(k) \ldots\ldots\ldots(30).$$

20. *To find the attraction parallel to the axis.*

Let Z, Z' be the z-components of the attraction; the surface conditions are

$$Z - Z' = 0,$$

and $$\frac{dZ}{d\nu} - \frac{dZ'}{d\nu} = -\frac{4\pi\sigma\gamma}{c^2}(2r^2 - b^2)\, z \ldots\ldots\ldots(31).$$

Now $$r^3 z = \frac{c^4 \mu\gamma}{(1+\gamma^2-\mu^2)^{\frac{5}{2}}};$$

therefore if $$u = \frac{\mu\gamma}{(1+\gamma^2-\mu^2)^{\frac{3}{2}}},$$

$$r^3 z = \frac{c^4}{3\gamma^2}\left(u - \gamma\frac{du}{d\gamma}\right);$$

therefore $$(2r^3 - b^2)\, rz = -\frac{c^4}{3\gamma^2}\left(u + 2\gamma\frac{du}{d\gamma}\right)$$

$$= -\frac{c^4}{3\gamma^2}\Sigma (4n+3)(2n+1) H_n \{q_{2n+1}(\gamma) + 2\gamma q'_{2n+1}(\gamma)\} P_{2n+1}(\mu)$$

by (24).

Hence

$$Z = \frac{1}{r}\Sigma A_{2n+1}\frac{p_{2n+1}(\nu)}{p_{2n+1}(\gamma)} P_{2n+1}(\mu),$$

$$Z' = \frac{1}{r}\Sigma A_{2n+1}\frac{q_{2n+1}(\nu)}{q_{2n+1}(\gamma)} P_{2n+1}(\mu);$$

therefore

$$\frac{dZ}{d\nu} - \frac{dZ'}{d\nu} = \frac{1}{r}\Sigma A_{2n+1}\left(\frac{p'_{2n+1}(\gamma)}{p_{2n+1}(\gamma)} - \frac{q'_{2n+1}(\gamma)}{q_{2n+1}(\gamma)}\right) P_{2n+1}(\mu),$$

$$= \frac{1}{r\cosh^2\beta}\frac{A_{2n+1}P_{2n+1}(\mu)}{P_{2n+1}(\gamma)\, q_{2n+1}(\gamma)};$$

therefore

$$A_{2n+1} = \frac{4\pi\sigma c^2 \cosh^2\beta}{3\sinh\beta} \times (4n+3)(2n+1) H_n P_{2n+1}(\gamma)\{q_{2n+1}(\gamma) + 2\gamma q'_{2n+1}(\gamma)\}.$$

21. *To find the attraction perpendicular to the axis.*
If R, R' be the attractions

$$\frac{dR}{d\nu} - \frac{dR'}{d\nu} = -\frac{4\pi\sigma\gamma}{c^2}(2r^2 - a^2)\rho.$$

Now
$$r^3\rho = \frac{c^4\sqrt{(1+\gamma^2)}\sqrt{(1-\mu^2)}}{(1+\gamma^2-\mu^2)^{\frac{5}{2}}};$$

therefore if
$$v = \frac{\sqrt{(1+\gamma^2)}\sqrt{(1-\mu^2)}}{(1+\gamma^2-\mu^2)^{\frac{3}{2}}},$$

$$r^3\rho = \tfrac{1}{3}c^4\left(\frac{v}{1+\gamma^2} - \frac{1}{\gamma}\frac{dv}{d\gamma}\right);$$

therefore

$$(2r^2 - a^2)\, r\rho = -\tfrac{1}{3}c^4\left(\frac{v}{1+\gamma^2} + \frac{2}{\gamma}\frac{dv}{d\gamma}\right)$$

$$= \frac{c^4}{3\sinh\beta\cosh\beta}\Sigma(4n+3)(2n+1)H_n$$

$$\times\left\{2q_{2n+1}(\gamma) - \frac{\gamma q'_{2n+1}(\gamma)}{(2n+1)(2n+2)}\right\}T^1_{2n+1}(\mu),$$

by (26). Hence (omitting the affix 1),

$$R = \frac{1}{r}\Sigma A_{2n+1}\frac{\mathfrak{T}_{2n+1}(\nu)}{\mathfrak{T}_{2n+1}(\gamma)}T_{2n+1}(\mu),$$

$$R' = \frac{1}{r}\Sigma A_{2n+1}\frac{\mathfrak{t}_{2n+1}(\nu)}{\mathfrak{t}_{2n+1}(\gamma)}T_{2n+1}(\mu);$$

therefore

$$\frac{dR}{d\nu} - \frac{dR'}{d\nu} = \frac{1}{r}\Sigma A_{2n+1}\left\{\frac{\mathfrak{T}'_{2n+1}(\gamma)}{\mathfrak{T}_{2n+1}(\gamma)} - \frac{\mathfrak{t}'_{2n+1}(\gamma)}{\mathfrak{t}_{2n+1}(\gamma)}\right\}T_{2n+1}(\mu).$$

But
$$\mathfrak{T}_n = \sqrt{(1+\gamma^2)}\,p'_n;$$

therefore
$$\mathfrak{T}'_n = \frac{n(n+1)p_n - \gamma p'_n}{\sqrt{(1+\gamma^2)}};$$

therefore
$$\mathfrak{T}'_n\mathfrak{t}_n - \mathfrak{T}_n\mathfrak{t}'_n = n(n+1)(p_n q'_n - p'_n q_n)$$

$$= -\frac{n(n+1)}{1+\gamma^2};$$

therefore

$$\frac{dR}{d\nu} - \frac{dR'}{d\nu} = -\frac{1}{r\cosh^2\beta}\,\Sigma\,\frac{(2n+1)(2n+2)A_{2n+1}T_{2n+1}(\mu)}{\mathfrak{T}_{2n+1}(\gamma)\,\mathfrak{t}_{2n+1}(\gamma)};$$

therefore

$$A_{2n+1} = \frac{4\pi\sigma c^2\cosh\beta}{3}\,\frac{4n+3}{2n+2}\,H_n\mathfrak{T}_{2n+1}(\gamma)\,\mathfrak{t}_{2n+1}(\gamma) \times\left\{2q_{2n+1}(\gamma) - \frac{\gamma q'_{2n+1}(\gamma)}{(2n+1)(2n+2)}\right\}$$

11, Old Square, Lincoln's Inn,
11*th July*, 1883.

ERRATA.

p. 359, line 6, read $\frac{P_n(\nu)}{P_n(\gamma)}$, for $\frac{P_n(\nu)}{P_n(\mu)}$.

p. 362, line 2 from bottom, read $\frac{d\psi}{d\eta}$ for $\frac{d\phi}{d\eta}$.

ON SOME ABELIAN INTEGRALS.

By Prof. H. J. R. Rink, Groningen, Holland.

A NOTE of Prof. Cayley in this Journal,* relating to the memoir of Captain MacMahon on the differential equation $X_1^{-\frac{3}{5}}dx_1 + X_2^{-\frac{3}{5}}dx_2 = 0$ has led me to examine the question whether other differential equations can also be integrated, by means of Abel's theorem.

The differential equation has the form

$$\frac{dx_1}{\sqrt[n]{\{R(x_1)^k\}}} + \frac{dx_2}{\sqrt[n]{\{R(x_2)^k\}}} = 0,$$

where

$$R(x)^k = (x-a_1)^{\mu_1}(x-a_2)^{\mu_2}(x-a_3)^{\mu_3}\ldots(x-a_m)^{\mu_m}.$$

In order that this equation may have an algebraic integral, it is necessary that $\int\frac{dx}{\sqrt[n]{\{R(x)^k\}}}$ be an integral of the first kind, and that the sum of an arbitrary number of such integrals be reduced by the Abel's theorem to a single integral.

In his famous memoir "Mémoire sur une propriété générale etc."† Abel has investigated, what is the smallest number of integrals to which the sum of an arbitrary number of similar integrals can be reduced. In our case, when the integral has the form $\int\frac{dx}{\sqrt[n]{\{R(x)^k\}}}$, this number p can be ex-

* Vol. XIX, p. 158.

† Oeuvres Complètes, Nouvelle Edition, t. 1, p. 145.

pressed by the formula

$$p = \tfrac{1}{2}\{(n-1)(m-1) - (\nu_0 - 1) - (\nu_1 - 1) - \ldots(\nu_m - 1)\},$$

where

ν_0 is the greatest common divisor of n and $\mu_1 + \mu_2 \ldots \mu_m$,

ν_1 " " " " " " n and μ_1,

ν_2 " " " " " " n and μ_2,

&c., &c., &c.

What are now the values of n, m, ν_0, $\nu_1 \ldots \nu_m$, for which p becomes $= 1$?

We suppose that n, μ_1, $\mu_2 \ldots \mu_m$ have no common divisor (except 1); and because the integrals are to be of the first kind we moreover suppose that μ_1, $\mu_2, \ldots \mu_m$ are each $< n$ and $\mu_1 + \mu_2 + \ldots \mu_m > n$.

We observe that the greatest value of ν_0 is n and the greatest value of any quantity $\nu_1, \nu_2 \ldots \nu_m$ is $\frac{1}{2}n$; but (except in the case $n = 2$) not all the quantities $\nu_1, \nu_2 \ldots \nu_m$ can have this greatest value; therefore (except in the case $n = 2$)

$$p > \tfrac{1}{2}\{(n-1)(m-1) - (n-1) - m(\tfrac{1}{2}n - 1)\}$$
$$> 1 + \tfrac{1}{2}n(m-4),$$

so that, in order that p may be $= 1$, m must be < 4. Only when $n = 2$, m can be $= 4$; and in this case

$$\nu_0 = 2,\ \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1.$$

We have now to consider the cases $m = 3$ and $m = 2$.

If $m = 3$, then

$$p = 1 + \tfrac{1}{2}(2n - \nu_0 - \nu_1 - \nu_2 - \nu_3),$$

and therefore $p = 1$, if $\nu_0 + \nu_1 + \nu_2 + \nu_3 = 2n$. When we again except the case $n = 2$, ν_0 must be $= n$ and $\nu_1 + \nu_2 + \nu_3$ must be $= n$, and it is easily proved that two of the quantities ν_1, ν_2, ν_3 cannot have the same value or a common divisor, except when they have the value 1. Hence we have $\nu_0 = 6$; $\nu_1, \nu_2, \nu_3 = 1, 2, 3$; $n = 6$.

The conclusion is, that if $m = 3$, we may have

$$n = 2,\ \nu_0 = 1,\ \nu_1 = 1,\ \nu_2 = 1,\ \nu_3 = 1,$$
$$n = 3,\ \nu_0 = 3,\ \nu_1 = 1,\ \nu_2 = 1,\ \nu_3 = 1,$$
$$n = 4,\ \nu_0 = 4,\ \nu_1 = 1,\ \nu_2 = 1,\ \nu_3 = 2,$$
$$n = 6,\ \nu_0 = 6,\ \nu_1 = 1,\ \nu_2 = 2,\ \nu_3 = 3.$$

By similar reasoning we find, that if $m = 2$, we may have

$$\begin{aligned} n = 3, \quad & \nu_0 = 1, \quad \nu_1 = 1, \quad \nu_2 = 1, \\ n = 4, \quad & \nu_0 = 2, \quad \nu_1 = 1, \quad \nu_2 = 1, \\ & \nu_0 = 1, \quad \nu_1 = 2, \quad \nu_2 = 1, \\ n = 6, \quad & \nu_0 = 3, \quad \nu_1 = 2, \quad \nu_2 = 1, \\ & \nu_0 = 2, \quad \nu_1 = 1, \quad \nu_2 = 3, \\ & \nu_0 = 1, \quad \nu_1 = 3, \quad \nu_2 = 2. \end{aligned}$$

By these considerations we find that only for the following differential expressions of the first kind and of the form $\frac{dx}{\sqrt[n]{\{R(x)^k\}}}$ the characteristic number p has the value 1.

$$\frac{dx}{\sqrt{\{(x-a)(x-b)(x-c)(x-d)\}}} \ldots(1), \qquad \frac{dx}{\sqrt[3]{[\{(x-a)(x-b)\}^2]}} \ldots(6),$$

$$\frac{dx}{\sqrt{\{(x-a)(x-b)(x-c)\}}} \ldots(2), \qquad \frac{dx}{\sqrt[4]{\{(x-a)^2(x-b)^3\}}} \ldots(7),$$

$$\frac{dx}{\sqrt[3]{[\{(x-a)(x-b)(x-c)\}^2]}} \ldots(3). \qquad \frac{dx}{\sqrt[4]{[\{(x-a)(x-b)\}^3]}} \ldots(8)$$

$$\frac{dx}{\sqrt[4]{\{(x-a)^2(x-b)^3(x-c)^3\}}} \ldots(4), \qquad \frac{dx}{\sqrt[6]{\{(x-a)^3(x-b)^4\}}} \ldots(9),$$

$$\frac{dx}{\sqrt[6]{\{(x-a)^3(x-b)^4(x-c)^5\}}} \ldots(5), \qquad \frac{dx}{\sqrt[6]{\{(x-a)^4(x-b)^5\}}} \ldots(10),$$

$$\frac{dx}{\sqrt[6]{\{(x-a)^4(x-b)^5\}}} \ldots(11).$$

We observe that the expressions (2), (6), (7), (8), (9), (10), (11) can be derived from the other by putting one of the quantities a, b, c, $d = \infty$. There remain to consider the expressions (1), (3), (4), (5). We only consider the expressions (4) and (5), for (1) is the elliptic differential expression and (3) that considered by Capt. MacMahon and Prof. Cayley.

The first equation which we will integrate is

$$\frac{dx_1}{\sqrt[4]{\{(x_1-a)^2(x_1-b)^3(x_1-c)^3\}}} + \frac{dx_2}{\sqrt[4]{\{(x_2-a)^2(x_2-b)^3(x_2-b)^3\}}} = 0.$$

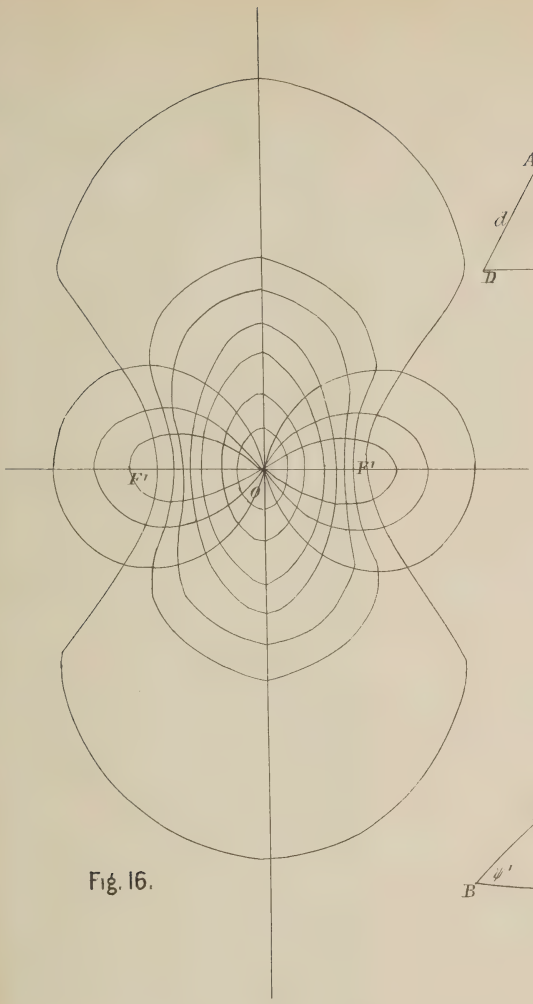


Fig. 16.

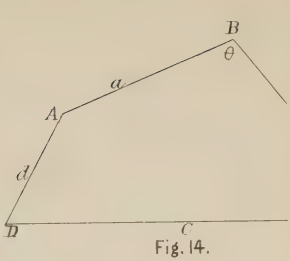


Fig. 14.

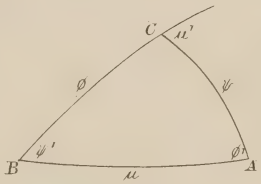


Fig. 15.

tho.

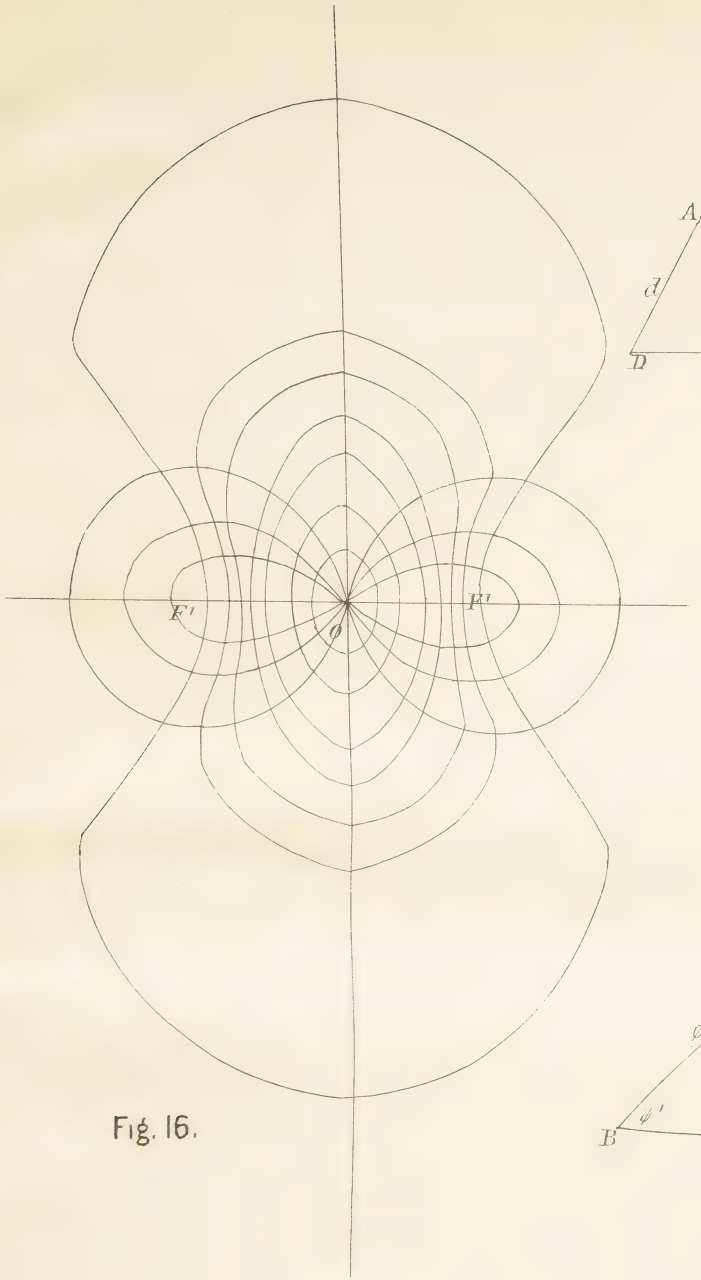


Fig. 16.

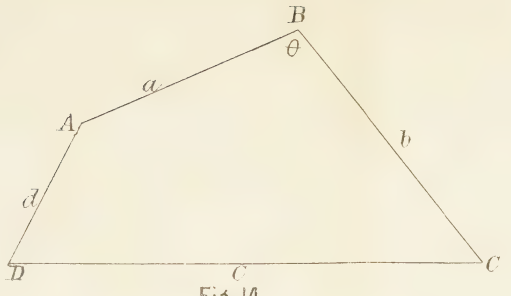


Fig. 14.

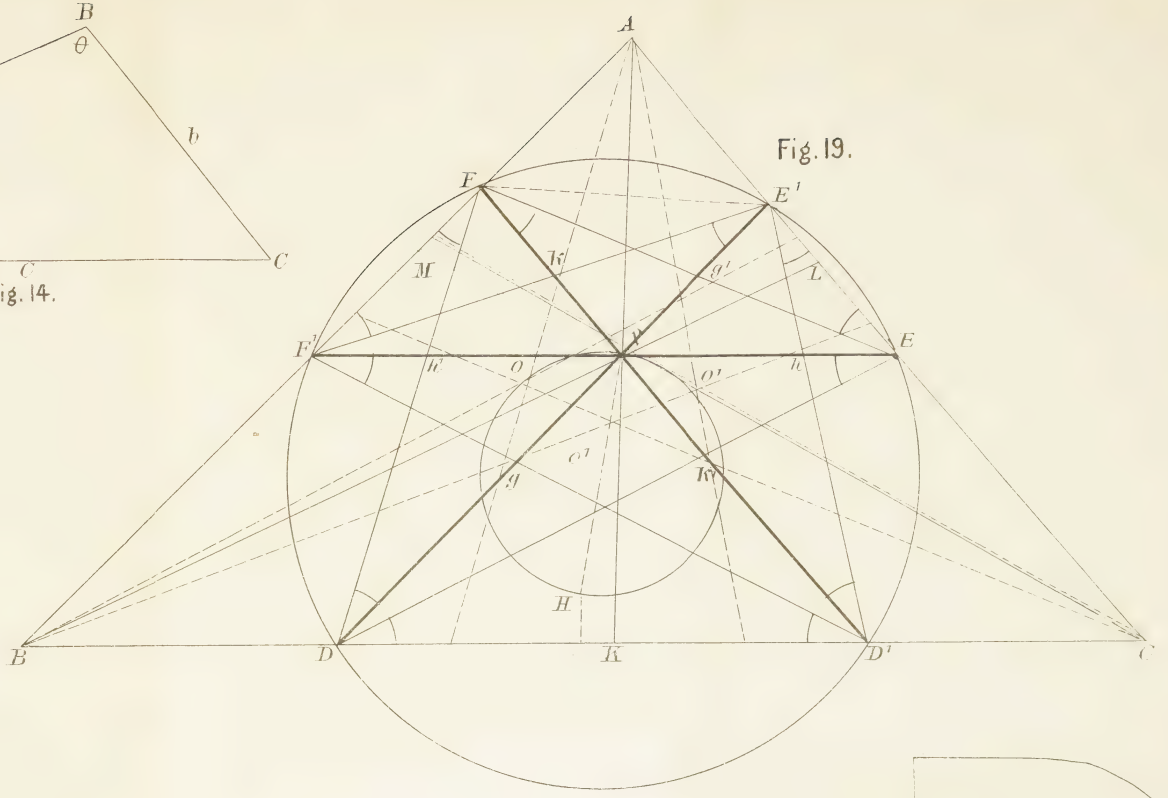


Fig. 19.

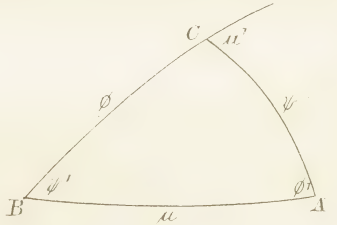


Fig. 15.

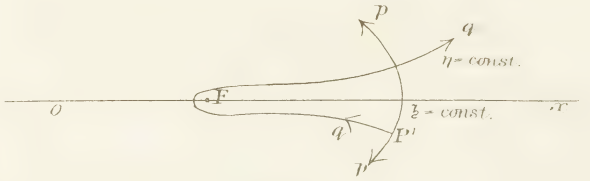


Fig. 17.

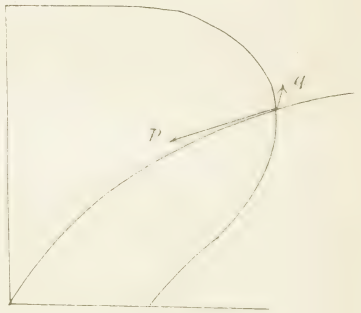


Fig. 18.

Lith. Anst. v. Werner & Winter, Frankfurt a. M.

UNIVERSITY OF ILLINOIS-URBANA
510.5QU C001
QUARTERLY JOURNAL OF PURE AND APPLIED MA
19 1883
3 0112 016836436